Solution (#1329) Let $a, b > 0$. Now, by the fundamental theorem of calculus,

$$\int_{x=0}^{x=\infty} e^{-xy}\,\mathrm{d}x = \left[\frac{e^{-xy}}{-y}\right]_0^\infty = \frac{1}{y},$$

and hence

$$\int_{y=a}^{y=b} \frac{1}{y}\,\mathrm{d}y = [\ln y]_a^b = \ln b - \ln a = \ln\left(\frac{b}{a}\right).$$

On the other hand, swapping the order of the integration, we have

$$\int_{y=a}^{y=b} e^{-xy}\,\mathrm{d}y = \left[\frac{e^{-xy}}{-x}\right]_a^b = \frac{e^{-ax} - e^{-bx}}{x}.$$

As

$$\int_{y=a}^{y=b}\left(\int_{x=0}^{x=\infty} e^{-xy}\,\mathrm{d}x\right)\mathrm{d}y = \int_{x=0}^{x=\infty}\left(\int_{y=a}^{y=b} e^{-xy}\,\mathrm{d}y\right)\mathrm{d}x$$

is given then

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x = \ln\left(\frac{b}{a}\right).$$