**Solution** (#1354) Let $z = \operatorname{cis}\theta$ so that $2\cos\theta = z + z^{-1}$. Then

$$\begin{aligned} 2^7\cos^7\theta &= \left(z+z^{-1}\right)^7 \\ &= \left(z^7+z^{-7}\right)+7\left(z^5+z^{-5}\right)+21(z^3+z^{-3})+35(z+z^{-1}) \\ &= 2\cos 7\theta+14\cos 5\theta+42\cos 3\theta+70\cos\theta. \end{aligned}$$

So

$$\cos^7\theta = \frac{1}{64}\left(\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\right)$$

and

$$\begin{aligned} \int \cos^7\theta \,\mathrm{d}\theta &= \frac{1}{64}\int\left(\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\right)\,\mathrm{d}\theta \\ &= \frac{1}{64}\left(\frac{\sin 7\theta}{7}+\frac{7}{5}\sin 5\theta+7\sin 3\theta+35\sin\theta\right)+\text{const.} \end{aligned}$$

To show that this agrees with the answer previous found in Example 5.32 we use De Moivre's theorem to note that

$$\sin n\theta = \operatorname{Im}\left(\cos\theta+i\sin\theta\right)^n$$

and so, writing $c=\cos\theta$ and $s=\sin\theta$, we have

$$\sin 3\theta = 3c^2s-s^3 = s\left(3c^2-(1-c^2)\right) = s(4c^2-1);$$

and

$$\begin{aligned} \sin 5\theta &= 5c^4s-10c^2s^3+s^5 \\ &= s\left(5c^4-10c^2\left(1-c^2\right)+\left(1-c^2\right)^2\right) \\ &= s\left(16c^4-12c^2+1\right); \end{aligned}$$

and

$$\begin{aligned} \sin 7\theta &= 7c^6s-35c^4s^3+21c^2s^5-s^7 \\ &= s\left(7c^6-35c^4\left(1-c^2\right)+21c^2\left(1-c^2\right)^2-\left(1-c^2\right)^3\right) \\ &= s\left(64c^6-80c^4+24c^2-1\right). \end{aligned}$$

So $\int\cos^7\theta\,\mathrm{d}\theta$ equals

$$\begin{aligned} &\frac{s}{64}\left(\frac{1}{7}\left(64c^6-80c^4+24c^2-1\right)+\frac{7}{5}\left(16c^4-12c^2+1\right)+7\left(4c^2-1\right)+35\right)+\text{const.} \\ =\ &\frac{c^6s}{7}+\frac{6c^4s}{35}+\frac{8c^2s}{35}+\frac{16s}{35}+\text{const.} \end{aligned}$$

which agrees with the answer from Example 5.32.