Solution (#1358) Let

$$I_n = \int_0^{\pi/2} x^n \sin x \, \mathrm{d}x.$$

Then

$$I_0 = \int_0^{\pi/2} \sin x \, \mathrm{d}x = [-\cos x]_0^{\pi/2} = 1,$$

and by IBP we have

$$\begin{aligned} I_1 &= \int_0^{\pi/2} x \sin x \, \mathrm{d}x \\ &= [-x\cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, \mathrm{d}x \\ &= [\sin x]_0^{\pi/2} \\ &= 1. \end{aligned}$$

For $n \geqslant 2$ we again have by IBP,

$$\begin{aligned} I_n &= \int_0^{\pi/2} x^n \sin x \, \mathrm{d}x \\ &= [-x^n \cos x]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1} (-\cos x) \, \mathrm{d}x \\ &= n \int_0^{\pi/2} x^{n-1} \cos x \, \mathrm{d}x \\ &= n \left\{ \left[x^{n-1} \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1) x^{n-2} \sin x \, \mathrm{d}x \right\} \\ &= n \left\{ \left(\frac{\pi}{2}\right)^{n-1} - (n-1) I_{n-2} \right\} \\ &= n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}. \end{aligned}$$

To conclude,

$$\begin{aligned} I_2 &= 2\left(\frac{\pi}{2}\right) - 2I_0 = \pi - 2; \\ I_3 &= 3\left(\frac{\pi}{2}\right)^2 - 6I_1 = \frac{3\pi^2}{4} - 6; \\ I_4 &= 4\left(\frac{\pi}{2}\right)^3 - 12I_2 = \frac{\pi^3}{2} - 12\pi + 24; \\ I_5 &= 5\left(\frac{\pi}{2}\right)^4 - 20I_3 = \frac{5\pi^4}{16} - 15\pi^2 + 120; \\ I_6 &= 6\left(\frac{\pi}{2}\right)^5 - 30I_4 = \frac{3\pi^5}{16} - 15\pi^3 + 360\pi - 720. \end{aligned}$$