Solution (#1359) For a natural number n let

$$I_n = \int_0^\infty x^n e^{-x^2} \, \mathrm{d}x.$$

When $n \geqslant 2$ we have by IBP that

$$\begin{aligned} I_n &= \int_0^\infty x^{n-1}(xe^{-x^2}) \, \mathrm{d}x \\ &= \left[x^{n-1}(-\frac{1}{2}e^{-x^2})\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}(\frac{-1}{2}e^{-x^2}) \, \mathrm{d}x \\ &= \left(\frac{n-1}{2}\right)\int_0^\infty x^{n-2}e^{-x^2} \, \mathrm{d}x \\ &= \left(\frac{n-1}{2}\right) I_{n-2}. \end{aligned}$$

By the FTC

$$I_1 = \int_0^\infty xe^{-x^2} \, \mathrm{d}x = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = \frac{1}{2}$$

and we are given that $I_0 = \sqrt{\pi}/2$. So if $n = 2k$ is even we have

$$\begin{aligned} I_{2k} &= \left(\frac{2k-1}{2}\right)\left(\frac{2k-3}{2}\right) \times \cdots \times \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\frac{\sqrt{\pi}}{2} \\ &= \frac{\sqrt{\pi}}{2^{k+1}} \frac{(2k)!}{[2k \times (2k-2) \times \cdots \times 2]} \\ &= \frac{\sqrt{\pi}}{2^{2k+1}} \frac{(2k)!}{k!}. \end{aligned}$$

And if $n = 2k+1$ then

$$I_{2k+1} = \left(\frac{2k}{2}\right)\left(\frac{2k-2}{2}\right) \times \cdots \times \left(\frac{4}{2}\right)\left(\frac{2}{2}\right)\frac{1}{2} = \frac{1}{2}k!.$$