Solution (#1361) Let m be a positive integer and α a positive real. For $m>1$ we have by IBP that

$$\begin{aligned} B(m,\alpha) &= \int_0^1 x^{m-1}(1-x)^{\alpha-1}\,\mathrm{d}x \\ &= \left[-\frac{1}{\alpha}x^{m-1}(1-x)^{\alpha}\,\mathrm{d}x\right]_0^1 + \frac{(m-1)}{\alpha}\int_0^1 x^{m-2}(1-x)^{\alpha}\,\mathrm{d}x \\ &= \frac{(m-1)}{\alpha}B(m-1,\alpha+1). \end{aligned}$$

If we repeatedly apply this recursion then

$$\begin{aligned} B(m,\alpha) &= \frac{(m-1)}{\alpha}B(m-1,\alpha+1) \\ &= \frac{(m-1)}{\alpha}\times\frac{(m-2)}{(\alpha+1)}\times B(m-2,\alpha+2) \\ &= \cdots \\ &= \frac{(m-1)(m-2)\times\cdots\times 1}{\alpha(\alpha+1)\times\cdots\times(\alpha+m-2)}B(1,m+\alpha-1). \end{aligned}$$

Now

$$B(1,m+\alpha-1) = \int_0^1 x^{1-1}(1-x)^{m+\alpha-2}\,\mathrm{d}x = \int_0^1 (1-x)^{m+\alpha-2}\,\mathrm{d}x = \frac{1}{m+\alpha-1}.$$

Hence

$$\begin{aligned} B(m,\alpha) &= \frac{(m-1)(m-2)\times\cdots\times 1}{\alpha(\alpha+1)\times\cdots\times(\alpha+m-2)(\alpha+m-1)} \\ &= \frac{(m-1)!}{\alpha(\alpha+1)\times\cdots\times(\alpha+m-2)(\alpha+m-1)} \end{aligned}$$

This can also be written as

$$B(m,\alpha) = \frac{(m-1)!\Gamma(\alpha)}{\Gamma(\alpha+m)}.$$

See more generally #1410.