Solution (#1365) For a natural number n, define

$$I_n = \int_1^\infty (x^2-1)^n e^{-x}\,\mathrm{d}x.$$

If $n \geqslant 1$, then by IBP we have

$$\begin{aligned} I_n &= \left[(x^2-1)^n(-e^{-x})\right] - \int_1^\infty 2nx(x^2-1)^{n-1}(-e^{-x})\,\mathrm{d}x \\ &= 2n\int_1^\infty x(x^2-1)^{n-1}e^{-x}\,\mathrm{d}x. \end{aligned}$$

Applying IBP again we find

$$\begin{aligned} \frac{I_n}{2n} &= \left[x(x^2-1)^{n-1}\left(-e^{-x}\right)\right]_1^\infty - \int_1^\infty \left\{(x^2-1)^{n-1} + 2(n-1)x^2(x^2-1)^{n-2}\right\}(-e^{-x})\,\mathrm{d}x \\ &= \int_1^\infty \left\{(x^2-1) + 2(n-1)x^2\right\}(x^2-1)^{n-2}e^{-x}\,\mathrm{d}x \\ &= \int_1^\infty \left\{(2n-1)(x^2-1) + (2n-2)\right\}(x^2-1)^{n-2}e^{-x}\,\mathrm{d}x \\ &= (2n-1)I_{n-1} + (2n-2)I_{n-2}. \end{aligned}$$

Now

$$I_0 = \int_1^\infty e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_1^\infty = \frac{1}{e},$$

and

$$\begin{aligned} I_1 &= \int_1^\infty (x^2-1)e^{-x}\,\mathrm{d}x \\ &= \left[(x^2-1)(e^{-x})\right]_1^\infty - \int_1^\infty 2x(-e^{-x})\,\mathrm{d}x \\ &= \int_1^\infty 2xe^{-x}\,\mathrm{d}x \\ &= \left[2x(-e^{-x})\right]_1^\infty + \int_1^\infty 2e^{-x}\,\mathrm{d}x \\ &= \frac{2}{e} + \frac{2}{e} = \frac{4}{e}. \end{aligned}$$

As it is true that eI_0 and eI_1 are positive integers it follows from the recursion

$$eI_n = 2n(2n-1)eI_{n-1} + 2n(2n-2)eI_{n-2}$$

that eI_n is a positive integer for all n.

To find I_5 specifically we see

$$\begin{aligned} I_2 &= 12I_1 + 8I_0 = (12\times 4 + 8\times 1)/e = 56/e; \\ I_3 &= 30I_2 + 24I_1 = (30\times 56 + 24\times 4)/e = 1776/e; \\ I_4 &= 56I_3 + 48I_2 = (56\times 1776 + 48\times 56)/e = 102144/e; \\ I_5 &= 90I_4 + 80I_3 = (90\times 102144 + 80\times 1776)/e = 9335040/e. \end{aligned}$$