Solution (#1367) For $n \geqslant 0$ define

$$I_n = \int_0^1 \frac{x^n}{1+x}\,\mathrm{d}x.$$

Note by the FTC that

$$I_0 = \int_0^1 \frac{\mathrm{d}x}{1+x} = [\ln(1+x)]_0^1 = \ln 2 - \ln 1 = \ln 2.$$

We also have that

$$\begin{aligned} I_n &= \int_0^1 \frac{x^n}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \frac{x^n + x^{n+1} - x^{n+1}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 x^n\,\mathrm{d}x - \int_0^1 \frac{x^{n+1}}{1+x}\,\mathrm{d}x \\ &= \frac{1}{n+1} - I_{n+1}. \end{aligned}$$

Hence

$$\begin{aligned} \ln 2 &= I_0 \\ &= 1 - I_1 \\ &= 1 - \frac{1}{2} + I_2 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - I_3 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n} + I_{2n}. \end{aligned}$$

As $(1+x)^{-1} \leqslant 1$ for $0 \leqslant x \leqslant 1$ then

$$0 \leqslant I_{2n} = \int_0^1 \frac{x^{2n}}{1+x}\,\mathrm{d}x \leqslant \int_0^1 x^{2n}\,\mathrm{d}x = \frac{1}{2n+1}.$$

As $n \to \infty$ then $I_{2n} \to 0$ and so in the limit

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots.$$