Solution (#1368) For $n \geqslant 0$ define

$$I_n = \int_0^1 \frac{x^n}{1+x^2}\,\mathrm{d}x.$$

Note by the FTC that

$$I_0 = \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

We also have that

$$\begin{aligned} I_n &= \int_0^1 \frac{x^n}{1+x^2}\,\mathrm{d}x \\ &= \int_0^1 \frac{x^n + x^{n+2} - x^{n+2}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 x^n\,\mathrm{d}x - \int_0^1 \frac{x^{n+2}}{1+x}\,\mathrm{d}x \\ &= \frac{1}{n+1} - I_{n+2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\pi}{4} &= I_0 \\ &= 1 - I_2 \\ &= 1 - \frac{1}{3} + I_4 \\ &= 1 - \frac{1}{3} + \frac{1}{5} - I_6 \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \cdots - \frac{1}{4n-1} + I_{4n}. \end{aligned}$$

As $\left(1+x^2\right)^{-1} \leqslant 1$ for $0 \leqslant x \leqslant 1$ then

$$0 \leqslant I_{4n} = \int_0^1 \frac{x^{4n}}{1+x^2}\,\mathrm{d}x \leqslant \int_0^1 x^{4n}\,\mathrm{d}x = \frac{1}{4n+1}.$$

As $n \to \infty$ then $I_{4n} \to 0$ and so in the limit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$