Solution (#1369) Let n be a natural number and

$$I_n = \int_0^1 (1-x)^n e^x \,\mathrm{d}x.$$

Then by IBP we find

$$I_n = \left[(1-x)^n e^x\right]_0^1 + \int_0^1 n(1-x)^{n-1} e^x \,\mathrm{d}x = -1 + nI_{n-1},$$

so that

$$I_{n-1} = \frac{1}{n} + \frac{1}{n} I_n.$$

Now

$$0 \leqslant I_n = \int_0^1 (1-x)^n e^x \,\mathrm{d}x \leqslant e \int_0^1 (1-x)^n \,\mathrm{d}x = \frac{e}{n+1}$$

and so $I_n \to 0$ as $n \to \infty$. Noting

$$I_0 = \int_0^1 e^x \,\mathrm{d}x = e - 1$$

we have

$$\begin{aligned}
e - 1 &= I_0 \\
&= 1 + I_1 \\
&= 1 + \frac{1}{2} + \frac{1}{2} I_2 \\
&= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{6} I_3 \\
&= 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} I_n \ () \\
&\to 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots
\end{aligned}$$

and the result follows.

Finally

$$e - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=n+1}^{\infty} \frac{1}{k!} > 0$$

and also

$$\begin{aligned}
e - \sum_{k=0}^{n} \frac{1}{k!} &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{n!} \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \right) \\
&< \frac{1}{n!} \left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \right) \\
&= \frac{1}{n!} \frac{1}{n+1} \bigg/ \left(1 - \frac{1}{n+1} \right) \\
&= \frac{1}{n!n}.
\end{aligned}$$

So if $e = m/n$ were rational then we would have

$$0 < \frac{m}{n} - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!n}$$

and if we multiply through by $n!$ then we find

$$0 < m(n-1)! - \sum_{k=0}^{n} \frac{n!}{k!} < \frac{1}{n} \leqslant 1.$$

The expression in the middle is necessarily an integer, but at the same time lies strictly between 0 and 1. This is the required contradiction and hence e is irrational.