Solution (#1377) (i) For $n \geqslant 2$, by IBP applied twice we have, that

$$
\begin{aligned}
I_n(x) &= x^{2n+1}\left\{\left[\left(1-t^2\right)^n \frac{\sin xt}{x}\right]_{-1}^{1} - \int_{-1}^{1} n(-2t)(1-t^2)^{n-1}\frac{\sin(xt)}{x}\,\mathrm{d}t\right\}\\
&= 2nx^{2n}\left\{\left[t(1-t^2)^{n-1}\left(\frac{-\cos(xt)}{x}\right)\right]_{-1}^{1} - \int_{-1}^{1}\left((1-t^2)^{n-1}-2(n-1)t^2(1-t^2)^{n-2}\right)\left(\frac{-\cos(xt)}{x}\right)\mathrm{d}t\right\}\\
&= 2nx^{2n-1}\left\{\int_{-1}^{1}\left((1-t^2)^{n-1}-2(n-1)t^2(1-t^2)^{n-2}\right)\cos(xt)\,\mathrm{d}t\right\}\\
&= 2nx^{2n-1}\left\{\int_{-1}^{1}\left((2n-1)(1-t^2)-2(n-1)\right)(1-t^2)^{n-2}\cos(xt)\,\mathrm{d}t\right\}\\
&= 2n(2n-1)I_{n-1}(x) - 4n(n-1)x^2 I_{n-2}(x)
\end{aligned}
$$

(ii) The first two integrals are

$$I_0(x) = \int_{-1}^{1} x\cos(xt)\,\mathrm{d}t = [\sin xt]_{-1}^{1} = 2\sin x$$

and by IBP

$$I_1(x) = x^3\int_{-1}^{1}(1-t^2)\cos(xt)\,\mathrm{d}t = -4x\cos x + 4\sin x.$$

As an inductive hypothesis, suppose that

$$I_n(x) = n!(P_n(x)\sin x + Q_n(x)\cos x)$$

where $P_n(x)$ and $Q_n(x)$ are polynomials in x of degree at most $2n$ and with integer coefficients. Note from the above that this is true of $n=0$ and $n=1$ with

$$P_0(x) = 2, \qquad Q_0(x) = 0, \qquad P_1(x) = 4, \qquad Q_1(x) = -4x.$$

If the hypothesis holds true for $n-1$ and $n-2$ then from the above reduction formula we have that $I_n(x)$ equals

$$
\begin{aligned}
&2n(2n-1)(n-1)!(P_{n-1}(x)\sin x + Q_{n-1}(x)\cos x) - 4n(n-1)x^2(n-2)!(P_{n-2}(x)\sin x + Q_{n-2}(x)\cos x)\\
=\;& n!\left\{\left[(4n-2)P_{n-1}(x) - 4x^2P_{n-2}(x)\right]\sin x + \left[(4n-2)Q_{n-1}(x) - 4x^2Q_{n-2}(x)\right]\cos x\right\}.
\end{aligned}
$$

Note that

$$P_n(x) = (4n-2)P_{n-1}(x) - 4x^2P_{n-2}(x), \qquad Q_n(x) = (4n-2)Q_{n-1}(x) - 4x^2Q_{n-2}(x)$$

are both integer-coefficient polynomials of degree less than or equal to $2n$ as P_{n-1}, Q_{n-1} had degree at most $2n-2$ and P_{n-2}, Q_{n-2} had degree at most $2n-4$.

(iii) We now set $x = \pi/2$ and suppose for a contradiction that $\pi/2 = a/b$ where a and b are natural numbers. Then

$$I_n(\pi/2) = \left(\frac{a}{b}\right)^{2n+1}\int_{-1}^{1}(1-t^2)^n\cos(\pi t/2)\,\mathrm{d}t = n!P_n\left(\frac{a}{b}\right)$$

which rearranges to

$$\frac{a^{2n+1}}{n!}\int_{-1}^{1}(1-t^2)^n\cos(\pi t/2)\,\mathrm{d}t = b^{2n+1}P_n\left(\frac{a}{b}\right).$$

As $P_n(x)$ has integer coefficients and degree at most $2n$ then the RHS is an integer. Note that the above integrand is positive and so the integral is also positive. Note further that

$$0 < \int_{-1}^{1}\left(1-t^2\right)^n\cos(\pi t/2)\,\mathrm{d}t \leqslant \int_{-1}^{1}\cos(\pi t/2)\,\mathrm{d}t = \frac{4}{\pi}.$$

Finally we note for any positive integer a that

$$0 < \frac{a^{2n+1}}{n!}\int_{-1}^{1}\left(1-t^2\right)^n\cos(\pi t/2)\,\mathrm{d}t \leqslant \frac{4a(a^2)^n}{\pi n!} \to 0$$

as $n \to \infty$ by #1287. In particular we can find an N such that

$$\frac{4a^{2N+1}}{\pi N!} < 1.$$

This means that there exists an integer strictly between 0 and 1 and this is the desired contradiction.