Solution (#1382) (i) By substituting $u = \sin x$ or by inspection we note

$$I_1 = \int_0^{\pi/2} \sin^{10} x \cos x \, \mathrm{d}x = \left[\frac{\sin^{11} x}{11} \right]_0^{\pi/2} = \frac{1}{11}.$$

(ii) Using $\sin^2 x = 1 - \cos^2 x$ we note

$$\begin{aligned} I_2 &= \int_0^{\pi/2} \cos^{11} x \sin^3 x \, \mathrm{d}x = \int_0^{\pi/2} \cos^{11} x \sin x - \cos^{13} x \sin x \, \mathrm{d}x \\ &= \left[-\frac{\cos^{12} x}{12} + \frac{\cos^{14} x}{14} \right]_0^{\pi/2} \\ &= \frac{1}{12} - \frac{1}{14} = \frac{1}{84}. \end{aligned}$$

(iii) As $\cos^4 x = (1 - \sin^2 x)^2 = 1 - 2\sin^2 x + \sin^4 x$ then

$$\begin{aligned} I_3 &= \int_0^{\pi/2} \sin^{11} x \cos^5 x \, \mathrm{d}x = \int_0^{\pi/2} (\sin^{11} x - 2\sin^{13} x + \sin^{15} x) \cos x \, \mathrm{d}x \\ &= \left[\frac{\sin^{12} x}{12} - 2 \times \frac{\sin^{14} x}{14} + \frac{\sin^{16} x}{16} \right]_0^{\pi/2} \\ &= \frac{1}{12} - \frac{1}{7} + \frac{1}{16} = \frac{1}{336}. \end{aligned}$$

(iv) As $2 \sin x \cos x = \sin 2x$ then

$$I_4 = \int_0^{\pi/2} \sin^6 x \cos^6 x \, \mathrm{d}x = \frac{1}{64} \int_0^{\pi/2} \sin^6 2x \, \mathrm{d}x.$$

We also have $2\sin^2 x = 1 - \cos 2x$ and $2\cos^2 x = 1 + \cos 2x$ then

$$\begin{aligned} I_4 &= \frac{1}{64} \times \frac{1}{8} \int_0^{\pi/2} (1 - \cos 4x)^3 \, \mathrm{d}x \\ &= \frac{1}{512} \int_0^{\pi/2} (1 - 3\cos 4x + 3\cos^2 4x - \cos^3 4x) \, \mathrm{d}x \\ &= \frac{1}{512} \int_0^{\pi/2} (1 - 3\cos 4x + \frac{3}{2}(1 + \cos 8x) - \cos 4x(1 - \sin^2 4x)) \, \mathrm{d}x \\ &= \frac{1}{512} \int_0^{\pi/2} (\frac{5}{2} - 4\cos 4x + \frac{3}{2} \cos 8x + \cos 4x \sin^2 4x) \, \mathrm{d}x \\ &= \frac{1}{512} \left[\frac{5x}{2} - \sin 4x + \frac{3}{16} \sin 8x + \frac{1}{12} \sin^3 4x \right]_0^{\pi/2} \\ &= \frac{5\pi}{2048}. \end{aligned}$$