Solution (#1385) Making the substitution $x = (\sqrt{2}\tan\theta - 1)/3$ so that

$$\mathrm{d}x = \frac{\sqrt{2}}{3}\sec^2\theta\,\mathrm{d}\theta,$$

we see

$$\begin{aligned}
\int \frac{\mathrm{d}x}{3x^2+2x+1} &= \frac{1}{3}\int \frac{\mathrm{d}x}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}} \\
&= \frac{1}{3}\int \frac{\frac{\sqrt{2}}{3}\sec^2\theta\,\mathrm{d}\theta}{\left(\frac{\sqrt{2}\tan\theta}{3}\right)^2+\frac{2}{9}} \\
&= \frac{1}{3}\int \frac{\frac{\sqrt{2}}{3}\sec^2\theta\,\mathrm{d}\theta}{\frac{2}{9}\left(\tan^2\theta+1\right)} \\
&= \frac{1}{3}\int \frac{\frac{\sqrt{2}}{3}\,\mathrm{d}\theta}{\frac{2}{9}} \\
&= \frac{1}{\sqrt{2}}\int \mathrm{d}\theta \\
&= \frac{\theta}{\sqrt{2}} + \text{const.} \\
&= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + \text{const.}.
\end{aligned}$$