Solution (#1386) Let n be a positive integer. Setting $x = \tan u$ in the given integral, we find

$$\begin{aligned} I_n &= \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^n} \\ &= \int_0^{\pi/2} \frac{\sec^2 u \,\mathrm{d}u}{\left(1+\tan^2 u\right)^n} \\ &= \int_0^{\pi/2} \frac{\sec^2 u \,\mathrm{d}u}{(\sec^2 u)^n} \\ &= \int_0^{\pi/2} \cos^{2n-2} u \,\mathrm{d}u. \end{aligned}$$

By #1353, with the given limits, and noting $I_1 = \pi/2$, we have for $n \geqslant 2$ that

$$\begin{aligned} I_n &= \frac{2n-3}{2n-2} I_{n-1} \\ &= \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2} \\ &= \frac{(2n)!}{2n(2n-1)\left[(2n-2)\times(2n-4)\times\cdots\times 2\right]} \times \frac{1}{2^{n-1}(n-1)!} \times \frac{\pi}{2} \\ &= \frac{(2n)!}{2n(2n-1)2^{n-1}(n-1)!} \times \frac{1}{2^{n-1}(n-1)!} \times \frac{\pi}{2} \\ &= \frac{(2n)!n\pi}{(2n-1)2^{2n}n!n!} \\ &= \frac{\pi n}{2^{2n}\,(2n-1)} \binom{2n}{n}. \end{aligned}$$