Solution (#1387) Let $0 < \alpha < \pi/2$. The sector bounded by the curve $r = R(\theta)$ and the half-lines $\theta = 0, \theta = \alpha$ can be divided into a triangle and remainder and so has area

$$A = \frac{1}{2} \times R(\alpha)\cos\alpha \times R(\alpha)\sin\alpha + \int_{x=R(\alpha)\cos\alpha}^{x=R(0)} f(x)\,\mathrm{d}x.$$

Now

$$f(x) = R(\theta)\sin\theta \qquad \text{and} \qquad x = R(\theta)\cos\theta.$$

Hence

$$\mathrm{d}x = \left(R'(\theta)\cos\theta - R(\theta)\sin\theta\right)\,\mathrm{d}\theta.$$

So

$$\begin{aligned}
\int_{x=R(\alpha)\cos\alpha}^{x=R(0)} f(x)\,\mathrm{d}x &= \int_{\theta=\alpha}^{\theta=0} R(\theta)\sin\theta\,\left(R'(\theta)\cos\theta - R(\theta)\sin\theta\right)\,\mathrm{d}\theta \\
&= \int_{\theta=\alpha}^{\theta=0} R(\theta)R'(\theta)\sin\theta\cos\theta\,\mathrm{d}\theta - \int_{\theta=\alpha}^{\theta=0} R(\theta)^2\sin^2\theta\,\mathrm{d}\theta.
\end{aligned}$$

Now applying IBP to the first integral we see

$$\begin{aligned}
\int_{\theta=\alpha}^{\theta=0} R(\theta)R'(\theta)\sin\theta\cos\theta\,\mathrm{d}\theta &= \left[\frac{R(\theta)^2}{2}\sin\theta\cos\theta\right]_{\alpha}^{0} - \int_{\theta=\alpha}^{\theta=0} \frac{R(\theta)^2}{2}\left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta \\
&= -\frac{R(\alpha)^2}{2}\sin\alpha\cos\alpha - \frac{1}{2}\int_{\theta=\alpha}^{\theta=0} R(\theta)^2\left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta.
\end{aligned}$$

Hence overall we have that A equals

$$\begin{aligned}
&\frac{R(\alpha)^2}{2}\sin\alpha\cos\alpha + \left\{-\frac{R(\alpha)^2}{2}\sin\alpha\cos\alpha - \frac{1}{2}\int_{\theta=\alpha}^{\theta=0} R(\theta)^2\left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta\right\} - \int_{\theta=\alpha}^{\theta=0} R(\theta)^2\sin^2\theta\,\mathrm{d}\theta \\
= &\ \frac{1}{2}\int_{\theta=0}^{\theta=\alpha} R(\theta)^2\left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta + \int_{\theta=0}^{\theta=\alpha} R(\theta)^2\sin^2\theta\,\mathrm{d}\theta \\
= &\ \frac{1}{2}\int_{\theta=0}^{\theta=\alpha} R(\theta)^2\left(\cos^2\theta + \sin^2\theta\right)\,\mathrm{d}\theta = \frac{1}{2}\int_{\theta=0}^{\theta=\alpha} R(\theta)^2\,\mathrm{d}\theta,
\end{aligned}$$

as required.