

Solution (#1388) Set $u = 1/x$ to see

$$\int_0^1 \frac{\ln x}{1+x^2} dx = - \int_1^\infty \frac{\ln u}{1+u^2} du$$

and hence

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = 0.$$

If we set $x = au$ in the integral $I(a)$ we find

$$I(a) = \frac{1}{a} \int_0^\infty \left(\frac{\ln a}{u^2+1} + \frac{\ln u}{u^2+1} \right) du = \frac{\pi \ln a}{2a}.$$