Solution (#1395) By setting $u = \pi/2 - x$ we can see that the first two integrals are equal. We see

$$\int_{z=0}^{z=\pi/2} \ln(\sin x)\,\mathrm{d}x = \int_{u=\pi 2}^{u=0} \ln(\sin(\pi/2 - u))\,(-\mathrm{d}u) = \int_{u=0}^{u=\pi/2} \ln(\cos u)\,\mathrm{d}u.$$

By the evenness of $\sin 2x$ about $x = \pi/4$ we set $u = 2x$ to see that

$$\int_{x=0}^{x=\pi/2} \ln(\sin 2x)\,\mathrm{d}x = \int_{x=0}^{x=\pi/4} \ln(\sin 2x)\;(2\mathrm{d}x) = \int_{u=0}^{u=\pi/2} \ln(\sin u)\,\mathrm{d}u.$$

Hence the three integrals are equal. Denote their common value as I. As $\sin 2x = 2\sin x \cos x$ then

$$\begin{aligned}
I &= \int_0^{\pi/2} \ln(\sin 2x)\,\mathrm{d}x \\
&= \int_0^{\pi/2} \ln(2\sin x \cos x)\,\mathrm{d}x \\
&= \int_0^{\pi/2} \ln 2\,\mathrm{d}x + \int_0^{\pi/2} \ln(\sin x)\,\mathrm{d}x + \int_0^{\pi/2} \ln(\cos x)\,\mathrm{d}x \\
&= \frac{\pi}{2}\ln 2 + 2I.
\end{aligned}$$

Hence $I = -\left(\pi/2\right)\ln 2$.

For the last two integrals note that substituting $x = \tan u$ in the first integral gives

$$\begin{aligned}
\int_0^{\infty} \frac{\ln(x^2+1)\,\mathrm{d}x}{x^2+1} &= \int_0^{\pi/2} \frac{\ln(\tan^2 u + 1)\;\sec^2 u\,\mathrm{d}u}{\tan^2 u + 1} \\
&= \int_0^{\pi/2} \ln(\sec^2 u)\,\mathrm{d}u \\
&= -2\int_0^{\pi/2} \ln(\cos u)\,\mathrm{d}u \\
&= -2\left(-\frac{\pi}{2}\ln 2\right) \\
&= \pi \ln 2.
\end{aligned}$$

For the final integral we will set $e^x = 1 + u^2$ so that

$$\int_0^{\infty} \frac{x}{\sqrt{e^x - 1}}\,\mathrm{d}x = \int_0^{\infty} \frac{\ln(1+u^2)}{u}\,\frac{2u\mathrm{d}u}{1+u^2} = 2\int_0^{\infty} \frac{\ln(1+u^2)}{1+u^2}\,\mathrm{d}u = 2\pi\ln 2.$$