Solution (#1403) Let $a, b > 0$ and set

$$I = \int_{-\infty}^{\infty} e^{-ax^2-b/x^2}\,\mathrm{d}x = 2\int_0^{\infty} e^{-ax^2-b/x^2}\,\mathrm{d}x$$

because of the evenness of the integrand. If we set $u = c/x$ for some $c > 0$ then we have

$$I = 2\int_{\infty}^{0} e^{-ac^2/u^2-bu^2/c^2}\left(-\frac{c\,\mathrm{d}u}{u^2}\right) = 2\int_0^{\infty} e^{-ac^2/u^2-bu^2/c^2}\,\frac{c\,\mathrm{d}u}{u^2}.$$

Note that if $c = \sqrt{b/a}$ then

$$-ac^2/u^2 - bu^2/c^2 = -au^2 - b/u^2.$$

Note also for this c that

$$-ax^2 - \frac{b}{x^2} = -a\left(x - \frac{c}{x}\right)^2 - 2ac$$

So for this choice of c we have

$$I = 2\int_0^{\infty} e^{-ax^2-b/x^2}\,\mathrm{d}x = 2\int_0^{\infty} e^{-au^2-b/u^2}\,\frac{c\,\mathrm{d}u}{u^2}.$$

Adding these expressions we see that

$$\begin{aligned} I &= \int_0^{\infty} e^{-ax^2-b/x^2}\left(1+\frac{c}{x^2}\right)\mathrm{d}x \\ &= \int_0^{\infty} \exp\left(-a\left(x-\frac{c}{x}\right)^2 - 2ac\right)\left(1+\frac{c}{x^2}\right)\mathrm{d}x \\ &= e^{-2ac}\int_0^{\infty} \exp\left(-a\left(x-\frac{c}{x}\right)^2\right)\left(1+\frac{c}{x^2}\right)\mathrm{d}x. \end{aligned}$$

Finally if we make the substitution

$$v = x - \frac{c}{x}$$

we see

$$I = e^{-2ac}\int_{-\infty}^{\infty} \exp\left(-av^2\right)\,\mathrm{d}v = e^{-2\sqrt{ab}}\sqrt{\frac{\pi}{a}}$$

by #1401.