Solution (#1410) We have been given that

$$B(a+1,-a) = \frac{-\pi}{\sin(\pi a)} \qquad \text{for } -1 < a < 0.$$

We also have from #1409 that

$$B(a+1,-a) = \int_0^\infty \frac{x^a}{1+x}\,\mathrm{d}x.$$

(i) Set $u = x^\alpha$ to find

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^\alpha} = \frac{\pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}.$$

(ii) Applying IBP, we have for $-1 < a < 0$ that

$$\frac{-\pi}{\sin(\pi a)} = \int_0^\infty \frac{x^a}{1+x}\,\mathrm{d}x = \left[\frac{x^{a+1}}{(a+1)(1+x)}\right]_0^\infty + \frac{1}{a+1}\int_0^\infty \frac{x^{a+1}}{(1+x)^2}\,\mathrm{d}x.$$

Hence for $-1 < a < 0$ we have

$$\int_0^\infty \frac{x^{a+1}}{(1+x)^2}\,\mathrm{d}x = \frac{-(a+1)\pi}{\sin(\pi a)}.$$

So for $0 < \beta = a+1 < 1$ we have

$$\int_0^\infty \frac{x^\beta}{(1+x)^2}\,\mathrm{d}x = \frac{-\beta\pi}{\sin(\pi(\beta-1))} = \frac{\pi\beta}{\sin(\pi\beta)}.$$

Now if $-1 < \beta < 0$ the substitution $u = 1/x$ gives

$$\int_0^\infty \frac{x^\beta}{(1+x)^2}\,\mathrm{d}x = \int_0^\infty \frac{u^{-\beta}}{(1+u)^2}\,\mathrm{d}u = \frac{\pi\beta}{\sin(\pi\beta)},$$

by the previous argument. If $\beta = 0$ then the given expression $\pi\beta/(\sin \pi\beta)$ is still valid if we understand it to represent the limit of 1 the expression has as $\beta \to 0$.

(iii) Argue carefully in a similar fashion to (ii) taking care of the separate cases $0 < \gamma < 2$ and $-1 < \gamma < 0$ to find

$$\int_0^\infty \frac{x^\gamma}{(1+x)^3}\,\mathrm{d}x = \frac{\pi\gamma(1-\gamma)}{2\sin(\pi\gamma))}$$

with the same limiting comments applying at $\gamma = 0$.

(iv) Set $u = x^2$ to find

$$\int_0^\infty \frac{x^\delta}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{u^{(\delta-1)/2}}{1+u}\,\mathrm{d}u \frac{\pi}{2\cos(\pi\delta/2)}.$$