Solution (#1411) Let $a, b > 0$ and recall that

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x.$$

If we set $x = u/(1+u)$ so that $\mathrm{d}x = \mathrm{d}u/(1+u)^2$ then

$$\begin{aligned} B(a,b) &= \int_{x=0}^{x=1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x \\ &= \int_{u=0}^{u=\infty} \left(\frac{u}{1+u}\right)^{a-1} \left(\frac{1}{1+u}\right)^{b-1} \frac{\mathrm{d}u}{(1+u)^2} \\ &= \int_{u=0}^{u=\infty} \frac{u^{a-1}}{(1+u)^{a+b}}\,\mathrm{d}u. \end{aligned}$$

Setting $u = (1+x)t$, we see for $x > 0$ that

$$\begin{aligned} \int_{t=0}^{t=\infty} e^{-(1+x)t}t^{a+b-1}\,\mathrm{d}t &= \int_{u=0}^{u=\infty} e^{-u}\left(\frac{u}{1+x}\right)^{a+b-1} \frac{\mathrm{d}u}{1+x} \\ &= \frac{1}{(1+x)^{a+b}}\int_{u=0}^{u=\infty} u^{a+b-1}e^{-u}\,\mathrm{d}u \\ &= \frac{\Gamma(a+b)}{(1+x)^{a+b}}. \end{aligned}$$

Finally we see

$$\begin{aligned} B(a,b) &= \int_{u=0}^{u=\infty} \frac{u^{a-1}}{(1+u)^{a+b}}\,\mathrm{d}u \\ &= \int_{u=0}^{u=\infty} u^{a-1}\left(\frac{1}{\Gamma(a+b)}\int_{t=0}^{t=\infty} e^{-(1+u)t}t^{a+b-1}\,\mathrm{d}t\right)\mathrm{d}u \\ &= \frac{1}{\Gamma(a+b)}\int_{u=0}^{u=\infty}\left(\int_{t=0}^{t=\infty} u^{a-1}t^{a+b-1}e^{-(1+u)t}\,\mathrm{d}t\right)\mathrm{d}u. \end{aligned}$$

If we now swap the order of integration we find

$$\begin{aligned} B(a,b) &= \frac{1}{\Gamma(a+b)}\int_{t=0}^{t=\infty}\left(\int_{u=0}^{u=\infty} u^{a-1}t^{a+b-1}e^{-(1+u)t}\,\mathrm{d}u\right)\mathrm{d}t \\ &= \frac{1}{\Gamma(a+b)}\int_{u=0}^{u=\infty} t^{b-1}e^{-t}\left(\int_{t=0}^{t=\infty}(ut)^{a-1}e^{-ut}\,t\mathrm{d}u\right)\mathrm{d}t \\ &= \frac{1}{\Gamma(a+b)}\int_{u=0}^{u=\infty} t^{b-1}e^{-t}\left(\int_{v=0}^{v=\infty} v^{a-1}e^{-v}\,\mathrm{d}v\right)\mathrm{d}t \qquad [v=ut] \\ &= \frac{1}{\Gamma(a+b)}\int_{u=0}^{u=\infty} t^{b-1}e^{-t}\Gamma(a)\,\mathrm{d}t \\ &= \frac{\Gamma(a)}{\Gamma(a+b)}\int_{u=0}^{u=\infty} t^{b-1}e^{-t}\,\mathrm{d}t \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned}$$