Solution (#1413) Let $x>0$. Setting $u=nt$ we find

$$B(x,n+1)n^x = n^x \int_{t=0}^{t=1} t^{x-1}(1-t)^n \,\mathrm{d}t = n^x \int_{u=0}^{u=n} \left(\frac{u}{n}\right)^{x-1}\left(1-\frac{u}{n}\right)^n \frac{\mathrm{d}u}{n} = \int_{u=0}^{u=n} u^{x-1}\left(1-\frac{u}{n}\right)^n \mathrm{d}u.$$

Now by #1331 we know

$$\int_{u=0}^{u=n} u^{x-1}\left(1-\frac{u}{n}\right)^n \mathrm{d}u \leqslant \int_{u=0}^{u=n} u^{x-1}e^{-u}\,\mathrm{d}u \leqslant \int_{u=0}^{u=\infty} u^{x-1}e^{-u}\,\mathrm{d}u = \Gamma(x).$$

Consider the integrals

$$I_{m,n} = \int_{u=0}^{u=n} u^{x-1}\left(1-\frac{u}{m}\right)^m \mathrm{d}u$$

where $0<n\leqslant m$. From #1331 and #1311 we have for $0\leqslant u\leqslant n<m$ that

$$0 \leqslant -u-\ln\left(1-\frac{u}{m}\right)^m \leqslant \frac{u^2}{m-u}$$

and so as exp is increasing

$$\exp\left(\frac{-mu}{m-n}\right) \leqslant \exp\left(\frac{-mu}{m-u}\right) \leqslant \left(1-\frac{u}{m}\right)^m \leqslant e^{-u}.$$

as $0\leqslant u\leqslant n$. So

$$\int_{u=0}^{u=n} u^{x-1}\exp\left(\frac{-mu}{m-n}\right)\mathrm{d}u \leqslant I_{m,n} \leqslant \int_{u=0}^{u=n} u^{x-1}e^{-u}\,\mathrm{d}u.$$

Looking at the LHS integral and setting $t=mu/(m-n)$ we see

$$\int_{u=0}^{u=n} u^{x-1}\exp\left(\frac{-mu}{m-n}\right)\mathrm{d}u = \left(\frac{m-n}{m}\right)^x \int_{t=0}^{t=mn/(m-n)} t^{x-1}\exp(-t)\,\mathrm{d}t.$$

So if we let $m\to\infty$ we see that

$$\lim_{m\to\infty} I_{m,n} = \int_{u=0}^{u=n} u^{x-1}e^{-u}\,\mathrm{d}u.$$

Then by definition $\lim_{n\to\infty}\lim_{m\to\infty} I_{m,n} = \Gamma(x)$. Provided $m\geqslant n$ then $I_{m,n}$ is increasing with both m and n and so

$$\lim_{n\to\infty}\int_{u=0}^{u=n} u^{x-1}\left(1-\frac{u}{n}\right)^n \mathrm{d}u = \lim_{n\to\infty} I_{n,n} = \lim_{n\to\infty}\lim_{m\to\infty} I_{m,n} = \Gamma(x).$$

From #1411 we know that

$$B(x,n+1)n^x = \frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)}n^x = \frac{n!\Gamma(x)n^x}{\Gamma(x+n+1)} = \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)}$$

as $\Gamma(x+n+1)=(x+n)\Gamma(x+n)=(x+n)(x+n-1)\cdots(x+1)\,x\Gamma(x)$. So

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)} \qquad \text{where } x>0. \tag{12.4}$$

Say now that $x<0$ and is not an integer and write $x=X+\varepsilon$ where $X=\lfloor x\rfloor$ and $0<\varepsilon<1$. Then by #1360

$$\Gamma(x)=\Gamma(X+\varepsilon)=\frac{\Gamma(X+1+\varepsilon)}{X+\varepsilon}=\frac{\Gamma(X+2+\varepsilon)}{(X+\varepsilon)(X+1+\varepsilon)}=\cdots=\frac{\Gamma(\varepsilon)}{(X+\varepsilon)(X+1+\varepsilon)\cdots(\varepsilon-2)(\varepsilon-1)}.$$

By (12.4) we have an expression for $\Gamma(\varepsilon)$ as a limit so that

$$\begin{aligned}
\Gamma(x) &= \frac{1}{(X+\varepsilon)(X+1+\varepsilon)\cdots(\varepsilon-2)(\varepsilon-1)} \times \lim_{n\to\infty}\frac{n!n^\varepsilon}{\varepsilon(\varepsilon+1)(\varepsilon+2)\cdots(\varepsilon+n)} \\
&= \lim_{n\to\infty}\frac{n!n^{x-X}}{x(x+1)\cdots(x+n-X)} \\
&= \lim_{n\to\infty}\frac{(N+X)!(N+X)^{x-X}}{x(x+1)\cdots(x+N)} \qquad [\text{setting } N=n-X] \\
&= \lim_{N\to\infty}\frac{N!N^x}{x(x+1)\cdots(x+N)} \times \lim_{N\to\infty}\frac{(N+X)!(N+X)^{-X}}{N!} \times \lim_{N\to\infty}\left(\frac{N+X}{N}\right)^x \\
&= \lim_{N\to\infty}\frac{N!N^x}{x(x+1)\cdots(x+N)} \times \lim_{N\to\infty}\frac{(N+X)^{-X}}{(N+X+1)(N+X+2)\times\cdots\times N} \times \lim_{N\to\infty}\left(\frac{N+X}{N}\right)^x \\
&= \lim_{N\to\infty}\frac{N!N^x}{x(x+1)\cdots(x+N)} \times 1 \times 1 = \lim_{N\to\infty}\frac{N!N^x}{x(x+1)\cdots(x+N)}.
\end{aligned}$$