Solution (#1420) Let $a \geqslant 0$. We begin with

$$\int_0^a \left(\int_0^\infty e^{-yx} \sin x \, \mathrm{d}x \right) \, \mathrm{d}y = \int_0^\infty \left(\int_0^a e^{-yx} \sin x \, \mathrm{d}y \right) \, \mathrm{d}x.$$

For $y > 0$, we have from #1344 that

$$\int_0^\infty e^{-yx} \sin x \, \mathrm{d}x = \frac{1}{1+y^2},$$

Hence recalling $\tan^{-1} y$ is an antiderivative of $(1+y^2)^{-1}$, we have

$$\int_0^a \left(\int_0^\infty e^{-yx} \sin x \, \mathrm{d}x \right) \, \mathrm{d}y = \tan^{-1} a.$$

We then note for $x > 0$ that

$$\int_0^a e^{-yx} \sin x \, \mathrm{d}y = \left[\frac{e^{-yx}}{-x} \sin x \right]_{x=0}^{x=a} = \frac{(1-e^{-ax})}{x} \sin x.$$

Hence we have

$$\int_0^\infty \left(1 - e^{-ax}\right) \frac{\sin x}{x} \, \mathrm{d}x = \tan^{-1} a.$$

If we let $a \to \infty$ then we see

$$\int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x = \lim_{a \to \infty} \tan^{-1} a = \frac{\pi}{2}.$$

Now let c be a real number. If $c = 0$ then

$$\int_0^\infty \frac{\sin cx}{x} \, \mathrm{d}x = 0.$$

If $c > 0$ then the substitution $u = xc$ gives

$$\int_{x=0}^{x=\infty} \frac{\sin cx}{x} \, \mathrm{d}x = \int_{u=0}^{u=\infty} \frac{\sin u}{u/c} \frac{\mathrm{d}u}{c} = \int_{u=0}^{u=\infty} \frac{\sin u}{u} \, \mathrm{d}u = \frac{\pi}{2}.$$

Finally if $c < 0$ then as sine is odd we have

$$\int_{x=0}^{x=\infty} \frac{\sin cx}{x} \, \mathrm{d}x = -\int_{x=0}^{x=\infty} \frac{\sin(-c)x}{x} \, \mathrm{d}x = -\frac{\pi}{2}.$$

Hence

$$\int_{x=0}^{x=\infty} \frac{\sin cx}{x} \, \mathrm{d}x = \frac{\pi}{2} \mathrm{sign}(c) = \begin{cases} \pi/2 & \text{if } c > 0; \\ 0 & \text{if } c = 0; \\ -\pi/2 & \text{if } c < 0. \end{cases}$$