Solution (#1422) Firstly note

$$\begin{aligned}
\int_{x=0}^{x=\infty} \frac{\mathrm{d}x}{(1+y)(1+x^2y)} &= \frac{1}{(1+y)} \int_{x=0}^{x=\infty} \frac{\mathrm{d}x}{(1+x^2y)} \\
&= \frac{1}{(1+y)} \int_{u=0}^{u=\pi/2} \frac{\frac{1}{\sqrt{y}} \sec^2 u \, \mathrm{d}u}{(1+\tan^2 u)} \qquad \left[x = \frac{1}{\sqrt{y}} \tan u\right] \\
&= \frac{\pi}{2\,(1+y)\,\sqrt{y}}.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\pi}{2} \int_{x=0}^{x=\infty} \frac{\mathrm{d}y}{(1+y)\sqrt{y}} &= \frac{\pi}{2} \int_{x=0}^{x=\infty} \frac{2t\,\mathrm{d}t}{(1+t^2)t} \qquad [y = t^2] \\
&= \pi \int_{x=0}^{x=\infty} \frac{\mathrm{d}t}{(1+t^2)} \\
&= \pi \left[\tan^{-1} t\right]_0^\infty \\
&= \frac{\pi^2}{2}.
\end{aligned}$$

Now swapping the order of integration as in #1329, we see by partial fractions that

$$\begin{aligned}
\int_{y=0}^{y=\infty} \frac{\mathrm{d}y}{(1+y)(1+x^2y)} &= \frac{1}{1-x^2} \int_{y=0}^{y=\infty} \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y} \right) \mathrm{d}y \\
&= \frac{1}{1-x^2} \left[\ln(1+y) - \ln\left(1+x^2y\right)\right]_0^\infty \\
&= \frac{1}{1-x^2} \left[\ln\left(\frac{1+y}{1+x^2y}\right)\right]_0^\infty \\
&= \frac{1}{1-x^2} \left(\ln x^{-2} - \ln 1\right) \\
&= \frac{2\ln x}{x^2-1}.
\end{aligned}$$

Hence, equating the two answers, we have

$$\int_{x=0}^{x=\infty} \frac{2\ln x}{x^2-1}\,\mathrm{d}x = \frac{\pi^2}{2}$$

and so

$$\int_{x=0}^{x=\infty} \frac{\ln x}{x^2-1}\,\mathrm{d}x = \frac{\pi^2}{4}.$$