Solution (#1426) Let $a, b > 0$. We will use the equality of the repeated integrals

$$\int_{y=0}^{y=b}\int_{x=0}^{x=\infty}\frac{(1-\cos ax)\cos yx}{x^2}\,\mathrm{d}x\,\mathrm{d}y=\int_{x=0}^{x=\infty}\int_{y=0}^{y=b}\frac{(1-\cos ax)\cos yx}{x^2}\,\mathrm{d}y\,\mathrm{d}x.$$

Evaluating the inner integral, we see the RHS equals

$$\int_0^\infty\frac{(1-\cos ax)}{x^2}\int_{y=0}^{y=b}\cos yx\,\mathrm{d}y\,\mathrm{d}x=\int_0^\infty\frac{(1-\cos ax)}{x^2}\left[\frac{\sin yx}{x}\right]_0^b\mathrm{d}x=\int_0^\infty\frac{(1-\cos ax)\sin bx}{x^3}\,\mathrm{d}x.$$

For $\alpha > 0$ we showed in #1424 that

$$\int_0^\infty\frac{1-\cos\alpha x}{x^2}\,\mathrm{d}x=\frac{\pi}{2}\alpha\qquad\text{so that for general }\alpha\text{ we have}\qquad\int_0^\infty\frac{1-\cos\alpha x}{x^2}\,\mathrm{d}x=\frac{\pi}{2}\left|\alpha\right|,$$

as cosine is even. Note also that $2\cos ax\cos yx=\cos((a+y)x)+\cos((a-y)x)$. Hence the LHS integral equals

$$\begin{aligned}&\frac{1}{2}\int_{y=0}^{y=b}\frac{2\cos yx-\cos((a+y)x)-\cos((a-y)x)}{x^2}\,\mathrm{d}y\\=&\frac{1}{2}\left\{\int_{y=0}^{y=b}\frac{1-\cos((a+y)x)}{x^2}\,\mathrm{d}y+\int_{y=0}^{y=b}\frac{1-\cos((a-y)x)}{x^2}\,\mathrm{d}y-2\int_{y=0}^{y=b}\frac{1-\cos yx}{x^2}\,\mathrm{d}y\right\}\\=&\frac{1}{2}\times\frac{\pi}{2}\times\{|a+y|+|a-y|-2|y|\}=\frac{\pi}{4}\{a-y+|a-y|\}.\end{aligned}$$

So

$$\int_{y=0}^{y=b}\int_{x=0}^{x=\infty}\frac{(1-\cos ax)\cos yx}{x^2}\,\mathrm{d}x\,\mathrm{d}y=\frac{\pi}{4}\int_{y=0}^{y=b}(|a+y|+|a-y|-2|y|)\,\mathrm{d}y.$$

We have two different cases to consider. Firstly let's assume $a > b$. Then

$$\frac{\pi}{4}\int_{y=0}^{y=b}a-y+|a-y|\,\mathrm{d}y=\frac{\pi}{4}\int_{y=0}^{y=b}2a-2y\,\mathrm{d}y=\frac{\pi}{4}\left[2ay-y^2\right]_0^b=\frac{1}{4}\pi b(2a-b).$$

If we have $a \leqslant b$ then

$$\frac{\pi}{4}\int_{y=0}^{y=b}a-y+|a-y|\,\mathrm{d}y=\frac{\pi}{4}\left\{\int_{y=0}^{y=a}2a-2y\,\mathrm{d}y+\int_{y=a}^{y=b}0\,\mathrm{d}y\right\}=\frac{1}{4}\pi a^2.$$

We've then shown

$$\int_0^\infty\frac{(1-\cos ax)\sin bx}{x^3}\,\mathrm{d}x=\begin{cases}\frac{1}{4}\pi a^2 & a\leqslant b\\ \frac{1}{4}\pi b(2a-b) & a>b\end{cases}.$$

Say now that $a \geqslant b > 0$ and $c > 0$. Note that

$$\int_0^\infty\frac{\sin ax\sin bx\sin cx}{x^3}\,\mathrm{d}x=\frac{1}{2}\int_0^\infty\frac{\{\cos(a-b)x-\cos(a+b)x\}\sin cx}{x^3}\,\mathrm{d}x$$

which we can rewrite as

$$\frac{1}{2}\left\{\int_0^\infty\frac{[1-\cos(a+b)x]\sin cx}{x^3}\,\mathrm{d}x-\int_0^\infty\frac{[1-\cos(a-b)x]\sin cx}{x^3}\,\mathrm{d}x\right\}.\tag{12.5}$$

If $a-b<a+b\leqslant c$ then (12.5) equates to

$$\frac{1}{2}\left\{\frac{\pi}{4}(a+b)^2-\frac{\pi}{4}(a-b)^2\right\}=\frac{1}{2}\pi ab.$$

If $a-b<c<a+b$ then (12.5) equates to

$$\frac{1}{2}\left\{\frac{\pi}{4}c(2(a+b)-c)-\frac{\pi}{4}(a-b)^2\right\}=\frac{\pi}{8}\left\{2ab+2ac+2bc-a^2-b^2-c^2\right\}.$$

If $c\leqslant a-b<a+b$ then (12.5) equates to

$$\frac{1}{2}\left\{\frac{\pi}{4}c(2(a+b)-c)-\frac{\pi}{4}c(2(a-b)-c)\right\}=\frac{1}{2}\pi bc.$$

In all we've shown

$$\int_0^\infty\frac{\sin ax\sin bx\sin cx}{x^3}\,\mathrm{d}x=\begin{cases}\frac{1}{2}\pi ab & a+b\leqslant c;\\ \frac{\pi}{8}\left\{2ab+2ac+2bc-a^2-b^2-c^2\right\} & a-b<c<a+b;\\ \frac{1}{2}\pi bc & c<a-b.\end{cases}$$