**Solution** (#1433) Note that

$$x^4 + a^4 = \left(x^2 - \sqrt{2}ax + a^2\right)\left(x^2 + \sqrt{2}ax + a^2\right),$$

and

$$\begin{aligned} \frac{1}{x^4+a^4} &= \frac{1}{2\sqrt{2}a^3}\left\{\frac{\sqrt{2}a - x}{x^2 - \sqrt{2}ax + a^2} + \frac{x+\sqrt{2}a}{x^2+\sqrt{2}ax+a^2}\right\}; \\ \frac{x^2}{x^4+a^4} &= \frac{1}{2\sqrt{2}a}\left\{\frac{x}{x^2 - \sqrt{2}ax + a^2} - \frac{x}{x^2+\sqrt{2}ax+a^2}\right\}. \end{aligned}$$

(i) and (ii) So

$$\int_0^\infty \frac{1}{x^4+a^4}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}a^3}. \qquad \int_0^\infty \frac{x^2}{x^4+a^4}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}a}.$$

(iii) Using the identities from (i) with $a=1$ we see

$$\int_0^1 \frac{\mathrm{d}x}{x^4+1} = \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right) + \frac{1}{2}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)\right]_0^1 = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2} + \ln(\sqrt{2}+1)\right].$$

(iv) And using the calculations from (ii) with $a=1$ we see

$$\int_0^1 \frac{x^2\,\mathrm{d}x}{x^4+1} = \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + \tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right)\right]_0^1 = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2} - \ln(\sqrt{2}+1)\right].$$

The first two integrals can be found from the integrals set in #1410 (but the last two cannot) by setting $x = au$ and $x^2 = a^2u$ respectively.