Solution (#1435) Note that

$$(3x+4)^2 = 9x^2+24x+16 = 9(x^2+2x+3)+2(3x+4)-19,$$

so we see

$$I_{m,n} = \int_{-\infty}^{\infty} \frac{\left[9(x^2+2x+3)+2(3x+4)-19\right](3x+4)^{m-2}}{(x^2+2x+3)^n}\,\mathrm{d}x = 9I_{m-2,n-1}+2I_{m-1,n}-19I_{m-2,n}$$

providing $m \geqslant 2$ and $n \geqslant 1$. So applying this recursion to $I_{3,4}$ we find

$$I_{3,4} = 9I_{1,3}+2I_{2,4}-19I_{1,4} \qquad \text{and} \qquad I_{2,4} = 9I_{0,3}+2I_{1,4}-19I_{0,4}$$

so that

$$I_{3,4} = 9I_{1,3}+18I_{0,3}+4I_{1,4}-38I_{0,4}-19I_{1,4} = 9I_{1,3}-15I_{1,4}+18I_{0,3}-38I_{0,4}.$$

Now

$$\begin{aligned}
I_{0,n} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+2x+3)^n} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(u^2+2)^n} \qquad [x=u-1] \\
&= 2^{1/2-n}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\sec^2\theta)^n} \qquad [u=\sqrt{2}\tan\theta] \\
&= 2^{3/2-n}\int_0^{\pi/2} \cos^{2n-2}\theta\,\mathrm{d}\theta \\
&= 2^{3/2-n}\binom{2n-2}{n-1}\frac{\pi}{2^{2n-1}} \qquad [\text{as found in \#263}] \\
&= \binom{2n-2}{n-1}\frac{\pi\sqrt{2}}{2^{3n-2}}.
\end{aligned}$$

And finally setting $u = x+1$ we see

$$I_{1,n}-I_{0,n} = \int_{-\infty}^{\infty} \frac{(3x+3)\,\mathrm{d}x}{(x^2+2x+3)^n} = \int_{-\infty}^{\infty} \frac{3u\,\mathrm{d}u}{(u^2+2)^n} = 0$$

as the integrand is odd. So we have

$$\begin{aligned}
I_{3,4} &= 9I_{1,3}-15I_{1,4}+18I_{0,3}-38I_{0,4} \\
&= 9I_{0,3}-15I_{0,4}+18I_{0,3}-38I_{0,4} \\
&= 27I_{0,3}-53I_{0,4} \\
&= 27\binom{4}{2}\frac{\pi\sqrt{2}}{2^7}-53\binom{6}{3}\frac{\pi\sqrt{2}}{2^{10}} \\
&= \frac{\pi\sqrt{2}}{2^{10}}\{27\times 6\times 8-53\times 20\} \\
&= \frac{\pi\sqrt{2}}{2^8}\{324-265\} \\
&= \frac{59\pi\sqrt{2}}{256}.
\end{aligned}$$