Solution (#1436) Let $0 < a < b$. We can rewrite our integrand as

$$\int_0^\infty \frac{\mathrm{d}x}{a + b\cosh x} = \int_0^\infty \frac{2e^x\,\mathrm{d}x}{be^{2x} + 2ae^x + b}.$$

If we make the substitution $u = e^x$ then the integral becomes

$$\int_1^\infty \frac{2\,\mathrm{d}u}{bu^2 + 2au + b} = \frac{2}{b}\int_1^\infty \frac{\mathrm{d}u}{u^2 + \frac{2a}{b}u + 1} = \frac{2}{b}\int_1^\infty \frac{\mathrm{d}u}{\left(u + \frac{a}{b}\right)^2 + \left(1 - \frac{a^2}{b^2}\right)}.$$

We will now make the substitution

$$u + \frac{a}{b} = \sqrt{1 - \frac{a^2}{b^2}}\tan\theta$$

so that the integral becomes

$$\frac{2}{b}\int_\phi^{\pi/2} \frac{\sqrt{1 - \frac{a^2}{b^2}}\sec^2\theta\,\mathrm{d}\theta}{\left(1 - \frac{a^2}{b^2}\right)\sec^2\theta}$$

where

$$\tan\phi = \frac{1 + \frac{a}{b}}{\sqrt{1 - \frac{a^2}{b^2}}} = \frac{b + a}{\sqrt{b^2 - a^2}} = \sqrt{\frac{b + a}{b - a}}.$$

The above integral equals

$$\frac{2}{b\sqrt{1 - \frac{a^2}{b^2}}}\int_\phi^{\pi/2} \mathrm{d}\theta = \frac{2}{\sqrt{b^2 - a^2}}\left(\frac{\pi}{2} - \phi\right) = \frac{2}{\sqrt{b^2 - a^2}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{b + a}{b - a}}\right).$$

Now for any $x > 0$, by considering the right-angled triangle with opposite x and adjacent 1, we see

$$\frac{\pi}{2} - \tan^{-1} x = \tan^{-1}\frac{1}{x}$$

so that our answer equals

$$\frac{2}{\sqrt{b^2 - a^2}}\tan^{-1}\left(\sqrt{\frac{b - a}{b + a}}\right)$$

as required.