Solution (#1441) Let $-1 < \alpha < 1$. Let $t = \tan(\theta/2)$, noting this to be strictly increasing for $-\pi < \theta < \pi$. Now

$$\cos\theta = \frac{1-t^2}{1+t^2}, \qquad d\theta = \frac{2dt}{1+t^2},$$

and so

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1+\alpha\cos\theta} &= \int_{-\pi}^{\pi} \frac{d\theta}{1+\alpha\cos\theta} \\
&= \int_{-\infty}^{\infty} \frac{\frac{2dt}{1+t^2}}{1+\alpha\left(\frac{1-t^2}{1+t^2}\right)} \\
&= 2\int_{-\infty}^{\infty} \frac{dt}{1+t^2+\alpha\,(1-t^2)} \\
&= 2\int_{-\infty}^{\infty} \frac{dt}{(1-\alpha)t^2+(1+\alpha)} \\
&= \frac{2}{1-\alpha}\int_{-\infty}^{\infty} \frac{dt}{t^2+(\frac{1+\alpha}{1-\alpha})}.
\end{aligned}$$

Now recall that an antiderivative of $\left(x^2+a^2\right)^{-1}$ is $\frac{1}{a}\tan^{-1}(x/a)$. So by the FTC

$$\begin{aligned}
\frac{2}{1-\alpha}\int_{-\infty}^{\infty} \frac{dt}{t^2+(\frac{1+\alpha}{1-\alpha})} &= \frac{2}{1-\alpha}\sqrt{\frac{1-\alpha}{1+\alpha}}\left[\tan^{-1}\left(\sqrt{\frac{1-\alpha}{1+\alpha}}t\right)\right]_{-\infty}^{\infty} \\
&= \frac{2}{\sqrt{1-\alpha}}\frac{1}{\sqrt{1+\alpha}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{2\pi}{\sqrt{1-\alpha^2}}.
\end{aligned}$$