Solution (#1454) Let $f(x) = ax^3 + bx^2 + cx + d$. Then on any interval $[x_{i-1}, x_{i+1}]$ we have

$$\int_{x_{i-1}}^{x_{i+1}} (ax^3 + bx^2 + cx + d)\,\mathrm{d}x = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + d\right]_{x_i-h}^{x_i+h}$$

which equals

$$\begin{aligned}
& \frac{a}{4}\left[(x_i+h)^4 - (x_i-h)^4\right] + \frac{b}{3}\left[(x_i+h)^3 - (x_i-h)^3\right] + \frac{c}{2}\left[(x_i+h)^2 - (x_i-h)^2\right] + d\left[(x_i+h) - (x_i-h)\right] \\
= \; & \frac{a}{4}\left[8x_i^3 h + 8x_i h^3\right] + \frac{b}{3}\left[6x_i^2 h + 2h^3\right] + \frac{c}{2}\left[4x_i h\right] + d[2h] \\
= \; & \frac{h}{3}\left(a\left[6x_i^3 + 6x_i h^2\right] + b\left[6x_i^2 + 2h^2\right] + 6cx_i + 6d\right) \\
= \; & \frac{h}{3}\left\{a\left[(x_i-h)^3 + 4x_i^3 + (x_i+h)^3\right] + b\left[(x_i-h)^2 + 4x_i^2 + (x_i+h)^2\right] + c\left[(x_i-h) + 4x_i + (x_i+h)\right] + d[1+4+1]\right\} \\
= \; & \frac{h}{3}\left[y_{i-1} + 4y_i + y_{i+1}\right].
\end{aligned}$$

Hence Simpson's rule estimates the definite integral exactly for cubics.

In fact there is no need to include the quadratic term in the above calculation in the above. It is sufficient to show that the above calculation works exactly for x^3 alone as Simpson's rule – by construction – will always perfectly adjust to the addition of a quadratic expression to any integrand. Simpson's rule works exactly for cubics because – even though the parabola does not exactly match the cubic's graph – an overestimate from one subinterval cancels out exactly the underestimate from the other.