Solution (#1455) Let $f(x)$ be a twice-differentiable function $f(x)$ defined on $[a,b]$. Then by IBP we have

$$\begin{aligned}\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x &= \left[(x-a)(x-b)f'(x)\right]_a^b - \int_a^b (2x-a-b)f'(x)\,\mathrm{d}x \\ &= -\int_a^b (2x-a-b)f'(x)\,\mathrm{d}x.\end{aligned}$$

Applying IBP once more we see that the above equals

$$-\left[(2x-a-b)f(x)\right]_a^b + \int_a^b 2f(x)\,\mathrm{d}x = -(b-a)(f(a)+f(b)) + \int_a^b 2f(x)\,\mathrm{d}x.$$

Rearranging the above calculation we get

$$\int_a^b f(x)\,\mathrm{d}x - \frac{(b-a)\left(f(b)+f(a)\right)}{2} = \frac{1}{2}\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x.$$

We also note at this stage that if $|f''(x)| \leqslant M$ for each x in $[a,b]$ then by #1277

$$\begin{aligned}\left|\frac{1}{2}\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x\right| &\leqslant \frac{M}{2}\int_a^b (x-a)(x-b)\,\mathrm{d}x \\ &= \frac{M}{2}\left[\frac{x^3}{3} - (a+b)\frac{x^2}{2} + abx\right]_a^b \\ &= \frac{M}{2}\left\{\frac{a^3}{6} - \frac{a^2b}{2} + \frac{ab^2}{2} - \frac{b^3}{6}\right\} \\ &= \frac{M}{12}(b-a)^3.\end{aligned}$$

We now apply the above calculations to each of the n subintervals of the trapezium rule. Let A_T denote the estimate made using the trapezium rule and E_T denote the error. Further say that $|f''(x)| \leqslant M$ for each x in $[a,b]$. Applying the above to the kth subinterval, where $k = 1,2,\ldots,n$ we find

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x - \frac{h}{2}(y_{k-1}+y_k) = \frac{1}{2}\int_{x_{k-1}}^{x_k} (x-x_{k-1})(x-x_k)f''(x)\,\mathrm{d}x$$

and we have by the second calculation that

$$\left|\frac{1}{2}\int_{x_{k-1}}^{x_k} (x-x_{k-1})(x-x_k)f''(x)\,\mathrm{d}x\right| \leqslant \frac{Mh^3}{12} = \frac{M(b-a)^3}{12n^3}.$$

Hence

$$E_T = \left|\int_a^b f(x)\,\mathrm{d}x - A_T\right| = \left|\sum_{k=1}^n \left(\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x - \frac{h}{2}(y_{k-1}+y_k)\right)\right| \leqslant n \times \frac{M(b-a)^3}{12n^3} = \frac{M(b-a)^3}{12n^2}.$$