Solution (#1466) A point P with position (X_t, Y_t) at time t, on a two-dimensional random walk of the integer grid $\{(x,y) : x, y \in \mathbb{Z}\}$ which begins with $X_0 = Y_0 = 0$. Let $0 < a, b, c, d < 1$ with $a+b+c+d = 1$. If $(X_t, Y_t) = (x, y)$ then (X_{t+1}, Y_{t+1}) equals one of

$$(x+1, y), \qquad (x-1, y), \qquad (x, y+1), \qquad (x, y-1),$$

with respective probabilities a, b, c, d.

(i) Let p_t denote the probability that $(X_t, Y_t) = (0,0)$. For (X_t, Y_t) to be back at the origin there must have been as many moves to the left as to the right, and as many moves up as down. So it must be the case that t is even.

Say then that $t = 2n$. It is then the case that there may have been $2k$ moves left and right and $2n-2k$ moves up and down to bring the point back to the origin, where $0 \leqslant k \leqslant n$. So

$$p_{2n} = \sum_{k=0}^{n} \binom{2n}{k, k, n-k, n-k} a^k b^k c^{n-k} d^{n-k}.$$

(ii) Now say that $a = b = c = d = 1/4$. The above becomes

$$p_{2n} = \sum_{k=0}^{n} \binom{2n}{k, k, n-k, n-k} \frac{1}{4^{2n}} = \frac{1}{4^{2n}} \sum_{k=0}^{n} \binom{2n}{k, k, n-k, n-k} = \frac{1}{4^{2n}} \binom{2n}{n}^2$$

by #318. By #278 we know that

$$\binom{2n}{n} \geqslant \frac{2^{2n}}{2\sqrt{n}}$$

and so

$$p_{2n} \geqslant \frac{1}{4^{2n}} \left(\frac{2^{2n}}{2\sqrt{n}} \right)^2 = \frac{1}{4n}$$

and $\sum_{t=0}^{\infty} p_t$ increases without bound by comparison with the harmonic series.

(iii) If it is the case that $a \neq b$ ignoring any up/down movements means we are back to a one-dimensional random walk in which the probabilities of left and right movements differ. From #1465 we know that the expected number of returns to the starting point is finite. For (X_t, Y_t) to be back at the origin it is necessarily the case that $X_t = 0$ and so we see $\sum_{t=0}^{\infty} p_t$ is finite when $a \neq b$ or $c \neq d$.