Solution (#1473) If we split the integral on $(0,2\pi)$ to two on $(0,\pi)$ and $(\pi,2\pi)$ and make the $t=\tan(x/2)$ substitution and recall that

$$\sin x=\frac{2t}{1+t^2},\qquad \cos t=\frac{1-t^2}{1+t^2},\qquad \mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2},$$

then we find

$$\begin{aligned}
\int_0^{2\pi}\frac{\sin^2 x}{2+\cos x}\,\mathrm{d}x &= \int_0^{\pi}\frac{\sin^2 x}{2+\cos x}\,\mathrm{d}x+\int_\pi^{2\pi}\frac{\sin^2 x}{2+\cos x}\,\mathrm{d}x\\
&= \int_0^{\infty}\frac{\left(\frac{2t}{1+t^2}\right)^2}{2+\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2\,\mathrm{d}t}{1+t^2}\right)+\int_{-\infty}^{0}\frac{\left(\frac{2t}{1+t^2}\right)^2}{2+\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2\,\mathrm{d}t}{1+t^2}\right)\\
&= 8\int_{-\infty}^{\infty}\frac{\left(\frac{t}{1+t^2}\right)^2}{2(1+t^2)+(1-t^2)}\,\mathrm{d}t\\
&= 8\int_{-\infty}^{\infty}\frac{t^2}{(1+t^2)^2(3+t^2)}\,\mathrm{d}t\\
&= 8\int_{-\infty}^{\infty}\left\{-\frac{3}{4(3+t^2)}+\frac{3}{4(1+t^2)}-\frac{1}{2(1+t^2)^2}\right\}\,\mathrm{d}t.
\end{aligned}$$

Now if we set $t=\sqrt{3}\tan\theta$ in the first integral and $t=\tan\theta$ in the second and third integrals, then

$$\begin{aligned}
\int_0^{2\pi}\frac{\sin^2 x}{2+\cos x}\,\mathrm{d}x &= 8\int_{-\pi/2}^{\pi/2}\left\{-\frac{3\sqrt{3}\sec^2\theta}{4(3\sec^2\theta)}+\frac{3\sec^2\theta}{4\sec^2\theta}-\frac{\sec^2\theta}{2(\sec^2\theta)^2}\right\}\,\mathrm{d}\theta\\
&= \int_{-\pi/2}^{\pi/2}\left\{-2\sqrt{3}+6-4\cos^2\theta\right\}\,\mathrm{d}\theta\\
&= (6-2\sqrt{3})\pi-\int_{-\pi/2}^{\pi/2}(2+2\cos 2\theta)\,\mathrm{d}\theta\\
&= (6-2\sqrt{3})\pi-\left[2\theta+\sin 2\theta\right]_{-\pi/2}^{\pi/2}\\
&= (6-2\sqrt{3})\pi-2\pi\\
&= \left(4-2\sqrt{3}\right)\pi.
\end{aligned}$$