Solution (#1477) Using $\cos 2x = 2\cos^2 x - 1$ we have

$$\int_0^{\pi/3} \sqrt{\cos x + \cos 2x}\,\mathrm{d}x = \int_0^{\pi/3} \sqrt{\cos x + 2\cos^2 x - 1}\,\mathrm{d}x = \int_0^{\pi/3} \sqrt{(\cos x + 1)(2\cos x - 1)}\,\mathrm{d}x.$$

If we similarly use $\cos x = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2)$ then the above integral becomes

$$\int_0^{\pi/3} \sqrt{2\cos^2(x/2)(1 - 4\sin^2(x/2)}\,\mathrm{d}x = \sqrt{2}\int_0^{\pi/3} \sqrt{1 - 4\sin^2(x/2)}\,\cos(x/2)\mathrm{d}x.$$

Now if we set $u = \sin(x/2)$ we find

$$\sqrt{2}\int_0^{\pi/3} \sqrt{1 - 4\sin^2(x/2)}\,\cos(x/2)\mathrm{d}x = \sqrt{2}\int_0^{1/2} \sqrt{1 - 4u^2}\,2\mathrm{d}u.$$

Finally setting $u = \frac{1}{2}\sin t$ we arrive at

$$\begin{aligned}
2\sqrt{2}\int_0^{1/2} \sqrt{1 - 4u^2}\,\mathrm{d}u &= 2\sqrt{2}\int_0^{\pi/2} \sqrt{1 - \sin^2 t}\,\frac{1}{2}\cos t\,\mathrm{d}t \\
&= \sqrt{2}\int_0^{\pi/2} \cos^2 t\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\int_0^{\pi/2} (1 + \cos 2t)\,\mathrm{d}t \\
&= \left[\frac{t}{\sqrt{2}} + \frac{\sin 2t}{2\sqrt{2}}\right]_0^{\pi/2} \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$