Solution (#1485) Note

$$\begin{aligned}\int_0^\infty e^{-ax^2}\sinh bx\,\mathrm{d}x &= \frac{1}{2}\left(\int_0^\infty e^{-ax^2+bx}\,\mathrm{d}x-\int_0^\infty e^{-ax^2-bx}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(\int_0^\infty e^{-a(x^2-bx/a)}\,\mathrm{d}x-\int_0^\infty e^{-a(x^2+bx)}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(e^{b^2/(4a)}\int_0^\infty e^{-a(x-b/(2a))^2}\,\mathrm{d}x-e^{b^2/(4a)}\int_0^\infty e^{-a(x+b/(2a))^2}\,\mathrm{d}x\right).\end{aligned}$$

If we set

$$u=\sqrt{a}\left(x-\frac{b}{2a}\right),\qquad v=\sqrt{a}\left(x+\frac{b}{2a}\right),$$

respectively into the first and second integrals then we find

$$\int_0^\infty e^{-ax^x}\sinh bx\,\mathrm{d}x=\frac{1}{2}e^{b^2/(4a)}\left(\int_{-b/(2\sqrt{a})}^\infty e^{-u^2}\,\frac{\mathrm{d}u}{\sqrt{a}}-\int_{b/(2\sqrt{a})}^\infty e^{-v^2}\,\frac{\mathrm{d}v}{\sqrt{a}}\right).$$

Recall now that the error function $\operatorname{erf} x$ is defined with the following properties

$$\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t,\qquad \operatorname{erf}(\pm\infty)=\pm 1,\qquad \operatorname{erf}(-x)=-\operatorname{erf}(x).$$

So

$$\begin{aligned}\int_0^\infty e^{-ax^x}\sinh bx\,\mathrm{d}x &= \frac{1}{2\sqrt{a}}e^{b^2/(4a)}\left(\int_{-b/(2\sqrt{a})}^\infty e^{-u^2}\,\mathrm{d}u-\int_{b/(2\sqrt{a})}^\infty e^{-v^2}\,\mathrm{d}v\right)\\ &= \frac{1}{2\sqrt{a}}e^{b^2/(4a)}\left\{\frac{\sqrt{\pi}}{2}\left(1-\operatorname{erf}\left(\frac{-b}{2\sqrt{a}}\right)\right)-\frac{\sqrt{\pi}}{2}\left(1-\operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right)\right)\right\}\\ &= \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{b^2/(4a)}\operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right).\end{aligned}$$