Solution (#1498) If we set $x = \tan u$ then we find

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x &= \int_0^{\pi/4} \frac{\ln(1+\tan u)}{1+\tan^2 u}\sec^2 u\,\mathrm{d}u \\
&= \int_0^{\pi/4} \ln(1+\tan u)\,\mathrm{d}u \\
&= \int_0^{\pi/4} \ln(\sin u + \cos u) - \ln\cos u\,\mathrm{d}u \\
&= \int_0^{\pi/4} \ln(\sqrt{2}\cos(u-\pi/4)) - \ln\cos u\,\mathrm{d}u \\
&= \int_0^{\pi/4} \frac{1}{2}\ln 2 + \ln\cos(u-\pi/4) - \ln\cos u\,\mathrm{d}u \\
&= \frac{\pi}{8}\ln 2 + \int_{-\pi/4}^{0} \ln\cos t\,\mathrm{d}t - \int_0^{\pi/4} \ln\cos u\,\mathrm{d}u \qquad [t = u - \pi/4] \\
&= \frac{\pi}{8}\ln 2
\end{aligned}$$

as cosine is even.