Solution (#1499) Let n be a natural number. Define

$$\begin{aligned} I_n &= \int_0^1 \frac{x^n}{\sqrt{x-x^2}}\,\mathrm{d}x \\ &= \int_0^1 x^{n-1/2}(1-x)^{-1/2}\,\mathrm{d}x \\ &= B(n+1/2,1/2) \\ &= \frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)} \qquad \text{by \#1410.} \end{aligned}$$

Now $\Gamma(1/2)=\sqrt{\pi}$ by #1401 and so by #1360

$$\begin{aligned} \Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \cdots \\ &= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\ &= \frac{(2n-1)(2n-3)\times\cdots\times 1}{2^n}\sqrt{\pi} \\ &= \frac{(2n)!}{(2n)(2n-2)\times\cdots\times 2}\frac{\sqrt{\pi}}{2^n} \\ &= \frac{(2n)!}{2^n n!}\frac{\sqrt{\pi}}{2^n}. \end{aligned}$$

Hence

$$I_n = \left(\frac{(2n)!}{2^n n!}\frac{\sqrt{\pi}}{2^n}\right)\frac{\sqrt{\pi}}{n!} = \binom{2n}{n}\frac{\pi}{2^{2n}}.$$