Solution (#1518) (i) The gamma distribution, with parameters $a > 0, \lambda > 0$, has pdf

$$\frac{1}{\Gamma(a)}\lambda^a x^{a-1} e^{-\lambda x} \mathbf{1}_{(0,\infty)}(x).$$

The expectation is then

$$\begin{aligned}
& \int_0^\infty x \times \frac{1}{\Gamma(a)}\lambda^a x^{a-1} e^{-\lambda x}\,\mathrm{d}x \\
= \; & \frac{1}{\Gamma(a)} \int_0^\infty \lambda^a x^a e^{-\lambda x}\,\mathrm{d}x \\
= \; & \frac{1}{\Gamma(a)} \int_0^\infty u^a e^{-u}\,\frac{\mathrm{d}u}{\lambda} \qquad [u = \lambda x] \\
= \; & \frac{1}{\lambda}\frac{\Gamma(a+1)}{\Gamma(a)} \\
= \; & \frac{a}{\lambda} \qquad \text{by \#1360.}
\end{aligned}$$

The variance equals

$$\begin{aligned}
& \int_0^\infty x^2 \times \frac{1}{\Gamma(a)}\lambda^a x^{a-1} e^{-\lambda x}\,\mathrm{d}x - \left(\frac{a}{\lambda}\right)^2 \\
= \; & \frac{1}{\Gamma(a)} \int_0^\infty \lambda^a x^{a+1} e^{-\lambda x}\,\mathrm{d}x - \left(\frac{a}{\lambda}\right)^2 \\
= \; & \frac{1}{\Gamma(a)} \int_0^\infty u^{a+1} e^{-u}\,\frac{\mathrm{d}u}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2 \qquad [u = \lambda x] \\
= \; & \frac{\Gamma(a+2)}{\Gamma(a)\lambda^2} - \left(\frac{a}{\lambda}\right)^2 \\
= \; & \frac{(a+1)a}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2 \\
= \; & \frac{a}{\lambda^2}.
\end{aligned}$$

(ii) Let X be a normally distributed random variable with $\mu = 0$ and $\sigma^2 = 1$. Let $Z = X^2$. If $z < 0$ then $P(Z \leqslant z) = 0$. If $z > 0$ then

$$F_Z(z) = P(Z \leqslant z) = P(-\sqrt{z} \leqslant X \leqslant \sqrt{z}) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{z}}^{\sqrt{z}} \exp\left(-\frac{t^2}{2}\right)\,\mathrm{d}t.$$

So, differentiating with respect to z using the chain rule, we have

$$\begin{aligned}
f_Z(z) &= \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{(\sqrt{z})^2}{2}\right) \times \frac{1}{2\sqrt{z}} - \exp\left(-\frac{(-\sqrt{z})^2}{2}\right) \times \frac{-1}{2\sqrt{z}} \right) \\
&= \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{z}{2}\right) \\
&= \frac{1}{\Gamma(1/2)} \left(\frac{1}{2}\right)^{1/2} z^{1/2-1} e^{-z/2}
\end{aligned}$$

as $\Gamma(1/2) = \sqrt{\pi}$ by #1401. The above is the pdf for the gamma distribution with $a = \lambda = 1/2$.