Solution (#1522) (i) Note that C_0 is an interval of length 1, and C_1 is 2 intervals of length 3^{-1}, and C_2 is 2^2 intervals of length 3^{-2}. More generally C_n will be 2^n intervals of length 3^{-n}. If true at the nth case then

$$\frac{1}{3}C_n \text{ consists of } 2^n \text{ intervals of length } 3^{-n-1} \quad \text{and} \quad \frac{1}{3}C_n + \frac{2}{3} \text{ consists of } 2^n \text{ intervals of length } 3^{-n-1},$$

importantly these latter intervals being disjoint from the former ones. So our claim follows by induction, showing

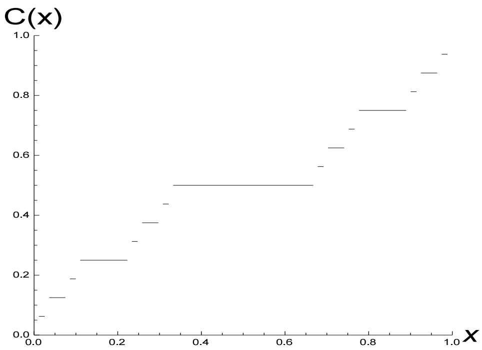
$$\int_0^1 \mathbf{1}_{C_n}(x)\,\mathrm{d}x = \left(\frac{2}{3}\right)^n.$$

The Cantor set C is the intersection of these C_n. The functions $\mathbf{1}_{C_n}(x)$ are step functions such that $0 \leqslant \mathbf{1}_C(x) \leqslant \mathbf{1}_{C_n}(x)$. As $(2/3)^n \to 0$ as $n \to \infty$ then it follows that the lower and upper Riemann integrals of $\mathbf{1}_C(x)$ are both 0. Hence $\mathbf{1}_C(x)$ is integrable and

$$\int_0^1 \mathbf{1}_C(x)\,\mathrm{d}x = 0.$$

(ii) Note that every number in the interval $(1/3, 2/3)$ has $n = 1$ and $k = 1$ so that $C(x) = 1/2$.
Every number in the interval $(1/9, 2/9)$ has $n = 2$ and $k = 1$ so that $C(x) = 1/4$.
Every number in the interval $(7/9, 8/9)$ has $n = 2$ and $k = 2$ so that $C(x) = 3/4$.
Every number in the interval $(1/27, 2/27)$ has $n = 3$ and $k = 1$ so that $C(x) = 1/8$.
Every number in the interval $(7/27, 8/27)$ has $n = 3$ and $k = 2$ so that $C(x) = 3/8$.
Every number in the interval $(19/27, 20/27)$ has $n = 3$ and $k = 3$ so that $C(x) = 5/8$.
Every number in the interval $(25/27, 26/27)$ has $n = 3$ and $k = 4$ so that $C(x) = 7/8$.

And so on, so that a graph of $C(x)$ looks like.



(iii) The integrals $\int \mathbf{1}_{C_n}(x)\,\mathrm{d}x$ converge to $\int \mathbf{1}_C(x)\,\mathrm{d}x$ as $\mathbf{1}_{C_n}(x)$ disagrees with $\mathbf{1}_C(x)$ by at most 1 on a set that becomes negligible in length. Considering the definition of C_n we see

$$\int \mathbf{1}_{C_n}(x)\,\mathrm{d}x = \frac{1}{3}\times\frac{1}{2} + \frac{1}{9}\times\left(\frac{1}{4}+\frac{3}{4}\right) + \frac{1}{27}\left(\frac{1}{8}+\frac{3}{8}+\frac{5}{8}+\frac{7}{8}\right) + \cdots + \frac{2^{n-2}}{3^n} = \frac{1}{6}\left(1+\frac{2}{3}+\cdots+\left(\frac{2}{3}\right)^{n-1}\right).$$

Taking the limit we have

$$\int \mathbf{1}_C(x)\,\mathrm{d}x = \frac{1/6}{1-2/3} = \frac{1/6}{1/3} = \frac{1}{2}.$$

Looking at the graph of $C(x)$ this should not be surprising given the rotational symmetry of the graph about $(1/2, 1/2)$.

(iv) Say for a contradiction that the Cantor distribution has a pdf $f(x)$. Take two points $x_1 < x_2$ which are both not in C_n for the same first time and are in the same subinterval of C_{n-1}. Then

$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x = C(x_2) - C(x_1) = 0.$$

As $f(x) \geqslant 0$ this means that $f(x) = 0$ at all points of the interval (x_1, x_2). This argument then shows that $f(x) = 0$ for all points not in C. At the same time we would also have that

$$\int_0^1 f(x)\,\mathrm{d}x = C(1) - C(0) = 1 - 0 = 1.$$

However as we saw from (i) we have that the integral of $f(x)\mathbf{1}_C(x)$ is 0. So we are left with a contradiction

$$1 = \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\mathbf{1}_C(x)\,\mathrm{d}x + \int_0^1 f(x)\mathbf{1}_{\text{complement of }C}(x)\,\mathrm{d}x = 0 + 0 = 0.$$