Solution (#1526) Recall that

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}).$$

If we define our inner product as

$$f \cdot g = \int_0^\infty f(x)g(x)e^{-x}\,\mathrm{d}x$$

then for $0 < n \leqslant m$ we have by IBP that

$$\begin{aligned} L_m \cdot L_n &= \frac{1}{m!}\int_0^\infty \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^m e^{-x})L_n(x)\,\mathrm{d}x \\ &= \frac{1}{m!}\left\{\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}(x^m e^{-x})L_n(x)\right]_0^\infty - \int_0^\infty \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}(x^m e^{-x})L_n'(x)\,\mathrm{d}x\right\} \\ &= \frac{-1}{m!}\int_0^\infty \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}(x^m e^{-x})L_n'(x)\,\mathrm{d}x. \end{aligned}$$

Applying IBP m times in all we find

$$L_m \cdot L_n = \frac{(-1)^m}{m!}\int_0^\infty x^m e^{-x}\frac{\mathrm{d}^m L_n}{\mathrm{d}x^m}\,\mathrm{d}x.$$

If we have that $m > n$ then $\mathrm{d}^m L_n/\mathrm{d}x^m = 0$ as L_n is a polynomial of degree n. Note that the leading coefficient of $L_n(x)$ is $(-1)^n/n!$ and so if we have $m = n$ then

$$\|L_n\|^2 = \frac{(-1)^n}{n!}\int_0^\infty x^n e^{-x}\frac{\mathrm{d}^n L_n}{\mathrm{d}x^n}\,\mathrm{d}x = \frac{(-1)^n}{n!}\int_0^\infty x^n e^{-x}(-1)^n\,\mathrm{d}x = \frac{1}{n!}\int_0^\infty x^n e^{-x}\,\mathrm{d}x = 1.$$

Finally if $0 = n \leqslant m$ then we have

$$L_m \cdot L_0 = \frac{1}{m!}\int_0^\infty \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^m e^{-x})\,\mathrm{d}x = 0 \qquad \text{when } m > 0$$

and

$$\|L_0\|^2 = \int_0^\infty e^{-x}\,\mathrm{d}x = 1.$$

So the Laguerre polynomials are orthogonal and are orthonormal with respect to the above inner product.