Solution (#1527) From #469(v) we know that

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})$$

and from (iii) that $H_n'(x) = 2nH_{n-1}(x)$.

If we take as our inner product

$$f \cdot g = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,\mathrm{d}x$$

then we have

$$H_m \cdot H_n = (-1)^n \int_{-\infty}^{\infty} H_m(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})\,\mathrm{d}x.$$

If $0 < m \leqslant n$ then by IBP we have

$$\begin{aligned} H_m \cdot H_n &= (-1)^n \left\{ \left[H_m(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(e^{-x^2}) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_m'(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(e^{-x^2})\,\mathrm{d}x \right\} \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} 2mH_{m-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(e^{-x^2})\,\mathrm{d}x \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} 2mH_{m-1}(x)\,(-1)^{n-1}\,H_{n-1}(x)e^{-x^2}\,\mathrm{d}x \\ &= 2mH_{m-1} \cdot H_{n-1}. \end{aligned}$$

So if $m \neq n$ then we have

$$2nH_{m-1} \cdot H_{n-1} = H_m \cdot H_n = 2mH_{m-1} \cdot H_{n-1}$$

and so $H_m \cdot H_n = 0$. If we have $m = n$ then

$$\|H_n\|^2 = 2n\,|H_{n-1}|^2 = 2^2 n(n-1)\,|H_{n-2}|^2 = 2^n n!\,|H_0|^2\,.$$

Finally for $0 = m < n$ we have

$$H_0 \cdot H_n = (-1)^n \int_{-\infty}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2})\,\mathrm{d}x = (-1)^n \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(e^{-x^2}) \right]_{-\infty}^{\infty} = 0$$

and when $m = n = 0$ we have

$$\|H_0\|^2 = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

Thus the Hermite polynomials are orthogonal with respect to the above inner product, but are not orthonormal and in fact

$$\|H_n\| = 2^{n/2}\sqrt{n!}\pi^{1/4}.$$