Solution (#1528) From #474 we have that

$$T_n(\cos\theta) = \cos n\theta.$$

So if $m \neq n$ then the substitution $x = \cos\theta$ leads to

$$T_m \cdot T_n = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{\pi}^{0} \frac{T_m(\cos\theta)T_n(\cos\theta)}{\sin\theta}\,(-\sin\theta\,\mathrm{d}\theta) = \int_0^{\pi} \cos m\theta\cos n\theta\,\mathrm{d}\theta.$$

We have the trigonometric identity

$$2\cos m\theta\cos n\theta = \cos(m-n)\theta + \cos(m+n)\theta$$

and so for $m \neq n$ we have

$$T_m \cdot T_n = \frac{1}{2}\left[\frac{\sin(m-n)\theta}{m-n} + \frac{\sin(m+n)\theta}{m+n}\right]_0^{\pi} = 0.$$

When $m = n$ we have

$$T_m \cdot T_n = \begin{cases} \pi & \text{when} \quad m = n = 0; \\ \frac{\pi}{2} & \text{when} \quad m = n > 0. \end{cases}$$

Hence the Chebyshev polynomials are orthogonal but not orthonormal with respect to this inner product.