Solution (#1530) Let n be a positive integer. Say that

$$x^n = \sum_{k=0}^{n} a_k T_k(x).$$

Using the inner product from #1528 we find

$$x^n \cdot T_k(x) = a_k \left\| T_k(x) \right\|^2.$$

So if $k \geqslant 1$ then

$$\begin{aligned} a_k &= \frac{x^n \cdot T_k(x)}{\left\| T_k(x) \right\|^2} \\ &= \frac{1}{\pi/2} \int_{-1}^{1} \frac{x^n T_k(x)}{\sqrt{1-x^2}}\, \mathrm{d}x \\ &= \frac{1}{\pi/2} \int_{\pi}^{0} \frac{\cos^n \theta T_k(\cos\theta)}{\sin\theta} (-\sin\theta\, \mathrm{d}\theta) \qquad [x = \cos\theta] \\ &= \frac{2}{\pi} \int_0^{\pi} \cos^n \theta \cos k\theta\, \mathrm{d}\theta. \end{aligned}$$

If $k = 0$ then we have

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \cos^n \theta\, \mathrm{d}\theta$$

as $\left\| T_0(x) \right\|^2 = \pi$.