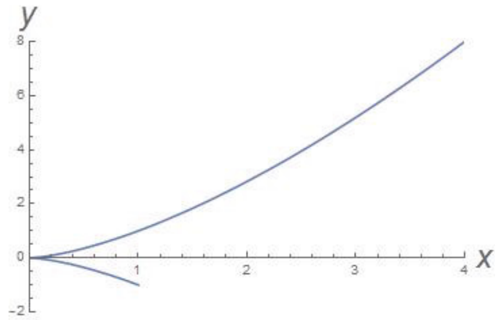
Solution (#1531) (i) We can parametrize a circle of radius a as $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta)$ where $0 \leqslant \theta \leqslant 2\pi$. Its length equals

$$\int_0^{2\pi} \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}\, \mathrm{d}\theta = \int_0^{2\pi} a\, \mathrm{d}\theta = 2\pi a.$$

(ii) A sketch of the curve $y^2 = x^3$ is given below.



Setting $f(x) = x^{3/2}$ we can find the arc length from $(0,0)$ to $(4,8)$ to equal

$$\int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, \mathrm{d}x = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^4 = \frac{8}{27}\left[10^{3/2} - 1\right].$$

In a similar fashion setting $f(x) = -x^{3/2}$ we can find the arc length from $(0,0)$ to $(1,-1)$ to equal

$$\int_0^1 \sqrt{1 + \left(-\frac{3}{2}x^{1/2}\right)^2}\, \mathrm{d}x = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^1 = \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right].$$

So the total arc length equals

$$\frac{8}{27}\left[10^{3/2} - 1\right] + \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right] = \frac{1}{27}\left(80\sqrt{10} + 13\sqrt{13} - 16\right).$$

(iii) If $x(t) = t^2$ and $y(t) = t^3$ with $-1 \leqslant t \leqslant 2$ then the arc length equals

$$\begin{aligned} &\int_{-1}^{2} \sqrt{(2t)^2 + (3t^2)^2}\, \mathrm{d}t = \int_{-1}^{2} |t| \sqrt{4 + 9t^2}\, \mathrm{d}t = -\int_{-1}^{0} t\sqrt{4+9t^2}\, \mathrm{d}t + \int_0^2 t\sqrt{4+9t^2}\, \mathrm{d}t \\ = &-\left[\frac{1}{27}(4+9t^2)^{3/2}\right]_{-1}^{0} + \left[\frac{1}{27}(4+9t^2)^{3/2}\right]_0^2 = \frac{1}{27}\left(80\sqrt{10} + 13\sqrt{13} - 16\right). \end{aligned}$$

(iv) Let $a > 0$. The curve $r = a(1+\cos\theta)$ can be parametrized by θ as

$$x = r\cos\theta = a(1+\cos\theta)\cos\theta; \qquad y = r\sin\theta = a(1+\cos\theta)\sin\theta,$$

and so its length equals

$$\int_0^{2\pi} \sqrt{a^2\left(-\sin\theta - 2\cos\theta\sin\theta\right)^2 + a^2(\cos\theta + \cos^2\theta - \sin^2\theta)^2}\, \mathrm{d}\theta.$$

Now

$$\begin{aligned} & \left(-\sin\theta - 2\cos\theta\sin\theta\right)^2 + (\cos\theta + \cos^2\theta - \sin^2\theta)^2 \\ = & \ (\sin\theta + \sin 2\theta)^2 + (\cos\theta + \cos 2\theta)^2 \\ = & \ \sin^2\theta + \cos^2\theta + \sin^2 2\theta + \cos^2 2\theta + 2\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta \\ = & \ 2 + 2\cos(2\theta - \theta) \\ = & \ 4\cos^2\frac{\theta}{2}. \end{aligned}$$

Hence the cardioid's length equals

$$\int_0^{2\pi} 2a\left|\cos\frac{\theta}{2}\right| \mathrm{d}\theta = 4a\int_0^{\pi} \cos\frac{\theta}{2}\, \mathrm{d}\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$