Solution (#1533) Let C be a curve parametrized separately as

$$\mathbf{r}(t) \qquad a \leqslant t \leqslant b; \qquad \mathbf{s}(u) \qquad \alpha \leqslant u \leqslant \beta$$

with the parametrizations connected by

$$\mathbf{r}(t) = \mathbf{s}(f(t)), \qquad f(a) = \alpha, \quad f(b) = \beta,$$

and then by the chain rule

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} f'(t).$$

Then using the definition of arc length based on the **r** parametrization we obtain

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t.$$

We now make the substitution $u = f(t)$ and the above becomes

$$\begin{aligned} & \int_{u=\alpha}^{u=\beta} \sqrt{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right] \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2} \, \frac{\mathrm{d}u}{f'(t)} \\ = & \int_{u=\alpha}^{u=\beta} \sqrt{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right] (f'(t))^2} \, \frac{\mathrm{d}u}{f'(t)} \\ = & \int_{u=\alpha}^{u=\beta} \sqrt{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2\right]} \, \mathrm{d}u \end{aligned}$$

as $f'(t) \geqslant 0$ for all t because f is increasing. This last integral is the arc length when calculated using the **s** parametrization and so arc length is independent of the choice of parametrization.