Solution (#1535) In a similar manner to the cylinder in #1534 we can wrap the plane onto the cone in a way that preserves lengths; this is identical to how one might use a sector of paper to make a conical hat or drinking cup. Explicitly this parametrization of the cone is

$$\mathbf{r}(r,\theta)=\left(\frac{r\cos(\sqrt{2}\theta)}{\sqrt{2}},\frac{r\sin(\sqrt{2}\theta)}{\sqrt{2}},\frac{r}{\sqrt{2}}\right),\qquad r>0,\ 0<\theta<\sqrt{2}\pi.$$

Note that a curve $(r(t)\cos\theta(t),r(t)\sin\theta(t))$ in the plane has length

$$\int_{t_1}^{t_2}\sqrt{\left(r'\cos\theta-r\sin\theta\theta'\right)^2+\left(r'\sin\theta+r\cos\theta\theta'\right)^2}\,\mathrm{d}t=\int_{t_1}^{t_2}\sqrt{(r')^2+(r\theta')^2}\,\mathrm{d}t.$$

On the cone we have

$$x'=\frac{1}{\sqrt{2}}r'\cos(\sqrt{2}\theta)-r\sin(\sqrt{2}\theta)\theta',\qquad y'=\frac{1}{\sqrt{2}}r'\sin(\sqrt{2}\theta)+r\cos(\sqrt{2}\theta)\theta',\qquad z'=\frac{r'}{\sqrt{2}}.$$

The corresponding curve on the cone has length

$$\begin{aligned}&\int_{t_1}^{t_2}\sqrt{\left[\frac{1}{\sqrt{2}}r'\cos(\sqrt{2}\theta)-r\sin(\sqrt{2}\theta)\theta'\right]^2+\left[\frac{1}{\sqrt{2}}r'\sin(\sqrt{2}\theta)+r\cos(\sqrt{2}\theta)\theta'\right]^2+\left[\frac{r'}{\sqrt{2}}\right]^2}\,dt\\ =\;&\int_{t_1}^{t_2}\sqrt{\left(\frac{r'}{\sqrt{2}}\right)^2+\left(r\theta'\right)^2+\left(\frac{r'}{\sqrt{2}}\right)^2}\,\mathrm{d}t=\int_{t_1}^{t_2}\sqrt{(r')^2+(r\theta')^2}\,\mathrm{d}t\end{aligned}$$

which is the same as that of the original curve.

(i) Note that $(1,0,1)$ and $(0,1,1)$ on the cone correspond to $(r,\theta)=\left(\sqrt{2},0\right)$ and $(r,\theta)=\left(\sqrt{2},\pi/\sqrt{8}\right)$ in the plane. These polar co-ordinates represent the actual points

$$\left(\sqrt{2},0\right)\qquad\text{and}\qquad\left(\sqrt{2}\cos\left(\frac{\pi}{\sqrt{8}}\right),\sqrt{2}\sin\left(\frac{\pi}{\sqrt{8}}\right)\right)$$

which are a distance

$$\begin{aligned}&\sqrt{\left(\sqrt{2}-\sqrt{2}\cos\left(\frac{\pi}{\sqrt{8}}\right)\right)^2+\left(\sqrt{2}\sin\left(\frac{\pi}{\sqrt{8}}\right)\right)^2}\\ =\;&2\sqrt{1-\cos\left(\frac{\pi}{\sqrt{8}}\right)}\\ =\;&2\sqrt{2}\sin\left(\frac{\pi}{\sqrt{32}}\right).\end{aligned}$$

(ii) Note that $(1,0,1)$ and $(0,2,2)$ on the cone correspond to $(r,\theta)=\left(\sqrt{2},0\right)$ and $(r,\theta)=\left(2\sqrt{2},\pi/\sqrt{8}\right)$ in the plane. These polar co-ordinates represent the actual points

$$\left(\sqrt{2},0\right)\qquad\text{and}\qquad\left(2\sqrt{2}\cos\left(\frac{\pi}{\sqrt{8}}\right),2\sqrt{2}\sin\left(\frac{\pi}{\sqrt{8}}\right)\right).$$

These are distance

$$\sqrt{\left(\sqrt{2}-2\sqrt{2}\cos\left(\frac{\pi}{\sqrt{8}}\right)\right)^2+\left(2\sqrt{2}\sin\left(\frac{\pi}{\sqrt{8}}\right)\right)^2}=\sqrt{2}\sqrt{5-4\cos\left(\frac{\pi}{\sqrt{8}}\right)}$$

apart in the plane and so the same distance apart on the cone.

(iii) As $(0,2,2)$ and $(1,0,-1)$ lie in two different halves of the cone, any curve between them must pass through $(0,0,0)$. The shortest distance from $(0,2,2)$ to $(0,0,0)$ within the cone is the straight line connecting them which has length $2\sqrt{2}$ and likewise the shortest distance from $(0,0,0)$ to $(1,0,-1)$ is the line segment between them which is of length $\sqrt{2}$. Hence the distance between them is $3\sqrt{2}$.