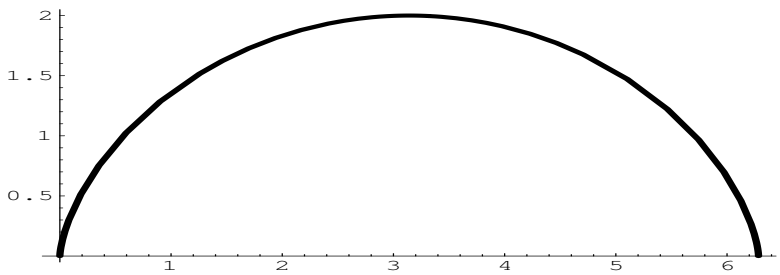
Solution (#1541) The cycloid is the curve given parametrically by the equations

$$x(t) = t - \sin t, \qquad \text{and} \qquad y(t) = 1 - \cos t \quad \text{for } 0 \leqslant t \leqslant 2\pi.$$

(i)



Sketch of the Cycloid

After the unit disc has rolled a distance of t then the centre of the disc is $(t, 1)$ and the fixed point has "moved back" so that radius from the fixed point to the centre now makes an angle of t. So the fixed point is at

$$(t, 1) + (-\sin t, -\cos t) = (t - \sin t, 1 - \cos t) = (x(t), y(t)).$$

(ii) Arc length $s(t)$ satisfies

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2 \frac{t}{2}.$$

Hence the cycloid's arc length equals

$$\int_0^{2\pi} \left|2\sin\frac{t}{2}\right| \, \mathrm{d}t = \left[-4\cos\frac{t}{2}\right]_0^{2\pi} = 8.$$

(iii) The area bounded between the cycloid and the x-axis equals

$$\begin{aligned}
\int_{t=0}^{t=2\pi} y \, \mathrm{d}x &= \int_0^{2\pi} y\frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, \mathrm{d}t \\
&= \left[\frac{3t}{2} - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi.
\end{aligned}$$

(iv) The area of the surface of revolution generated by the cycloid equals

$$\begin{aligned}
2\pi \int_{t=0}^{t=2\pi} y \, \mathrm{d}s &= 2\pi \int_0^{2\pi} y\frac{\mathrm{d}s}{\mathrm{d}t} \, \mathrm{d}t \\
&= 2\pi \int_0^{2\pi} (1 - \cos t) \times 2\sin\frac{t}{2} \, \mathrm{d}t \\
&= 2\pi \left[-8\cos\frac{t}{2} + \frac{8}{3}\cos^3\frac{t}{2}\right]_0^{2\pi} \\
&= 2\pi\left(8 - \frac{8}{3} + 8 - \frac{8}{3}\right) = \frac{64\pi}{3}.
\end{aligned}$$

(v) The volume enclosed by the surface of revolution generated by the cycloid equals

$$\begin{aligned}
\pi \int_{t=0}^{t=2\pi} y^2 \, \mathrm{d}x &= \pi \int_0^{2\pi} y^2\frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\
&= \pi \int_0^{2\pi} (1 - \cos t)^2(1 - \cos t) \, \mathrm{d}t \\
&= \pi \int_0^{2\pi} \left(\frac{5}{2} - \frac{15}{4}\cos t + \frac{3}{2}\cos 2t - \frac{1}{4}\cos 3t\right) \, \mathrm{d}t \\
&= \pi \left[\frac{5t}{2} - \frac{15}{4}\sin t + \frac{3}{4}\sin 2t - \frac{1}{2}\sin 3t\right]_0^{2\pi} = 5\pi^2
\end{aligned}$$