**Solution** (#1542) (i) An upside-down cycloid has parametrization

$$x(u) = u - \sin u, \qquad \text{and} \qquad y(u) = \cos u \qquad \text{for } 0 \leqslant u \leqslant \pi.$$

A smooth wire in the shape of this cycloid is fashioned, and a particle of mass $m$ is released from the point $(0,1)$. From conservation of energy we know that

$$\frac{1}{2}m\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right) + mgy = E = \text{constant}.$$

From the initial starting position, with the particle being at rest at $(0,1)$ we know that $E = mg$.

By the chain rule we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1-\cos u)\frac{\mathrm{d}u}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\sin u \frac{\mathrm{d}u}{\mathrm{d}t}$$

and so we have

$$\frac{m}{2}\left((1-\cos u)^2 + \sin^2 u\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + mg\cos u = mg$$

which rearranges to $(\mathrm{d}u/\mathrm{d}t)^2 = g$. So $\mathrm{d}u/\mathrm{d}t = \sqrt{g}$ and hence $u = \sqrt{g}t$. As $u = \pi$ at the bottom of the cycloid then the time taken is $\pi/\sqrt{g}$.

(ii) Arguing similarly when the particle starts at rest from $(x(u_0), y(u_0))$ we have

$$\frac{m}{2}\left((1-\cos u)^2 + \sin^2 u\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + mg\cos u = mg\cos u_0.$$

which rearranges to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{g}\sqrt{\frac{\cos u_0 - \cos u}{1-\cos u}}.$$

and the time taken to reach the bottom equals

$$T = \frac{1}{\sqrt{g}}\int_{u=u_0}^{u=\pi}\sqrt{\frac{1-\cos u}{\cos u_0 - \cos u}}\,\mathrm{d}u = \frac{1}{\sqrt{g}}\int_{u=u_0}^{u=\pi}\frac{\sin\frac{u}{2}}{\sqrt{\cos^2\frac{u_0}{2} - \cos^2\frac{u}{2}}}\,\mathrm{d}u.$$

In a similar fashion to #1552 we now make the substitution

$$\cos\frac{u}{2} = \cos\frac{u_0}{2}\cos\theta$$

so that $\frac{1}{2}\sin\frac{u}{2}\,\mathrm{d}u = \cos\frac{u_0}{2}\sin\theta\,\mathrm{d}\theta$. Then we have

$$T = \frac{1}{\sqrt{g}}\int_{u=0}^{u=\pi/2}\frac{2\cos\frac{u_0}{2}\sin\theta\,\mathrm{d}\theta}{\sqrt{\cos^2\frac{u_0}{2} - \cos^2\frac{u_0}{2}\cos^2\theta}} = \frac{1}{\sqrt{g}}\int_{u=0}^{u=\pi/2}\frac{2\sin\theta\,\mathrm{d}\theta}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{g}}\int_{u=0}^{u=\pi/2}2\,\mathrm{d}\theta = \frac{\pi}{\sqrt{g}}.$$

(iii) Say now that the particle travels down a smooth linear wire from $(0,1)$ to $(\pi,0)$. We can parametrize this as

$$x(u) = u, \qquad \text{and} \qquad y(u) = 1 - \frac{u}{\pi} \qquad \text{for } 0 \leqslant u \leqslant \pi.$$

By conservation of energy we again have

$$\frac{1}{2}m\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right) + mgy = mg$$

so that

$$\frac{1}{2}m\left(1+\frac{1}{\pi^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + mg\left(1-\frac{u}{\pi}\right) = mg$$

which rearranges to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{\frac{2g\pi u}{\pi^2+1}}.$$

This is a separable DE and we find that the time taken equals

$$T' = \sqrt{\frac{\pi^2+1}{2g\pi}}\int_{u=0}^{u=\pi}\sqrt{u}\,\mathrm{d}u = \sqrt{\frac{\pi^2+1}{2g\pi}}\left[\frac{2u^{3/2}}{3}\right]_0^\pi = \frac{2}{3}\pi\sqrt{\frac{\pi^2+1}{2g}}.$$

Now this time $T'$ is greater than the previous time $T$ as $\pi^2+1 > 9$ and so

$$T' = \frac{2}{3}\pi\sqrt{\frac{\pi^2+1}{2g}} > \frac{2}{3}\pi\sqrt{\frac{9}{2g}} = \frac{\sqrt{2}\pi}{\sqrt{g}} > \frac{\pi}{\sqrt{g}} = T.$$