Solution (#1547) Setting $x = 2\tan^2 t$ we find

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x(x+2)(x+3)}} &= \int_0^{\pi/2} \frac{4\tan t \sec^2 t\,\mathrm{d}t}{\sqrt{2\tan^2 t(2+2\tan^2 t)(3+2\tan^2 t)}} \\
&= \frac{4}{\sqrt{2}\sqrt{2}} \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{(3\cos^2 t + 2\sin^2 t)}} \\
&= 2\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{3-\sin^2 t}} \\
&= \frac{2}{\sqrt{3}} \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-\frac{1}{3}\sin^2 t}} \\
&= \frac{2}{\sqrt{3}} K\left(\frac{1}{\sqrt{3}}\right).
\end{aligned}$$

Again setting $x = 2\tan^2 t$ we find

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x(x^2+2x+4)}} &= \int_0^{\pi/2} \frac{4\tan t \sec^2 t\,\mathrm{d}t}{\sqrt{2\tan^2 t(4\tan^4 t + 4\tan^2 t + 4)}} \\
&= \sqrt{2}\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{\sin^4 t + \sin^2 t\cos^2 t + \cos^4 t}} \\
&= \sqrt{2}\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-\sin^2 t\cos^2 t}} \\
&= \sqrt{2}\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-\frac{1}{4}\sin^2 2t}} \\
&= \sqrt{2}\int_0^{\pi} \frac{\mathrm{d}u/2}{\sqrt{1-\frac{1}{4}\sin^2 u}} \qquad [u = 2t] \\
&= \sqrt{2}\int_0^{\pi/2} \frac{\mathrm{d}u}{\sqrt{1-\frac{1}{4}\sin^2 u}} \\
&= \sqrt{2}K\left(\frac{1}{2}\right).
\end{aligned}$$