Solution (#1549) Let $0 \leqslant k \leqslant 1$ and define $k' = \sqrt{1-k^2}$.

We make the substitution $\tan(x-t) = k'\tan t$ or equivalently

$$x = t + \tan^{-1}(k'\tan t).$$

Note that when $t=0$ then $x=0$ and that when $t=\pi/2$ then $x=\pi$. Also

$$\mathrm{d}x = \left(1 + \frac{k'\sec^2 t}{1+k'^2\tan^2 t}\right)\mathrm{d}t,$$

and note that

$$\frac{k'\sec^2 t}{1+k'^2\tan^2 t} = \frac{k'}{\cos^2 t + k'^2\sin^2 t} = \frac{k'}{1-k^2\sin^2 t}.$$

We further note that

$$\begin{aligned}
\sin x &= \sin(t+\tan^{-1}(k'\tan t)) \\
&= \sin t\cos\left[\tan^{-1}(k'\tan t)\right] + \cos t\sin\left[\tan^{-1}(k'\tan t)\right] \\
&= \frac{\sin t}{\sqrt{1+k'^2\tan^2 t}} + \frac{\cos t(k'\tan t)}{\sqrt{1+k'^2\tan^2 t}} \\
&= \frac{(1+k')\sin t}{\sqrt{1+k'^2\tan^2 t}} \\
&= \frac{(1+k')\sin t\cos t}{\sqrt{1-k^2\sin^2 t}}.
\end{aligned}$$

Hence

$$\begin{aligned}
K\left(\frac{1-k'}{1+k'}\right) &= \frac{1}{2}\int_0^{\pi}\frac{(1+k')\,\mathrm{d}x}{\sqrt{(1+k')^2-(1-k')^2\sin^2 x}} \\
&= \frac{1+k'}{2}\int_0^{\pi/2}\frac{1}{\sqrt{(1+k')^2-(1-k')^2\left[\frac{(1+k')^2\sin^2 t\cos^2 t}{1-k^2\sin^2 t}\right]}}\left(1+\frac{k'}{1-k^2\sin^2 t}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{\pi/2}\frac{\sqrt{1-k^2\sin^2 t}}{\sqrt{(1-k^2\sin^2 t)-(1-k')^2\sin^2 t\cos^2 t}}\left(1+\frac{k'}{1-k^2\sin^2 t}\right)\mathrm{d}t
\end{aligned}$$

Now

$$\begin{aligned}
&(1-k^2\sin^2 t)-(1-k')^2\sin^2 t\cos^2 t \\
&= 1-k^2\sin^2 t-(1-k')^2\sin^2 t(1-\sin^2 t) \\
&= 1-(k^2+(1-k')^2)\sin^2 t+(1-k')^2\sin^4 t \\
&= 1-2(1-k')\sin^2 t+(1-k')^2\sin^4 t \\
&= (1-(1-k')\sin^2 t)^2.
\end{aligned}$$

So

$$K\left(\frac{1-k'}{1+k'}\right) = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1+k'-k^2\sin^2 t}{1-(1-k')\sin^2 t}\right)\frac{\mathrm{d}t}{\sqrt{1-k^2\sin^2 t}}$$

Finally we note $k^2 = 1-k'^2$ and so

$$1+k'-k^2\sin^2 t = 1+k'-\left(1-k'^2\right)\sin^2 t = (1+k')(1-(1-k')\sin^2 t)$$

and hence

$$K\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}\int_0^{\pi/2}\frac{\mathrm{d}t}{\sqrt{1-k^2\sin^2 t}} = \left(\frac{1+k'}{2}\right)K(k).$$