Solution (#1551) For $a, b > 0$ define

$$I(a,b) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}}.$$

(i) If $a \geqslant b$ then

$$\begin{aligned} I(a,b) &= \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a^2(1-\sin^2 x) + b^2 \sin^2 x}} \\ &= \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a^2 - (a^2 - b^2)\sin^2 x}} \\ &= \frac{1}{a}\int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2 x}} \\ &= \frac{1}{a} K\left(\sqrt{1 - \frac{b^2}{a^2}}\right). \end{aligned}$$

(ii) By making the substitution $t = \pi/2 - x$ we see

$$I(a,b) = \int_{\pi/2}^0 \frac{-\mathrm{d}t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} = \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} = I(b,a).$$

Also for $c > 0$ we have

$$I(a,b) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{c^2 a^2 \cos^2 x + c^2 b^2 \sin^2 x}} = \frac{1}{c}\int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}} = \frac{I(a,b)}{c}.$$

Finally we note

$$I(a,a) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a^2\cos^2 x + a^2 \sin^2 x}} = \int_0^{\pi/2} \frac{\mathrm{d}x}{a} = \frac{\pi}{2a}.$$

(iii) Without loss of generality assume that $a \geqslant b$ so that $(a+b)/2 \geqslant \sqrt{ab}$ as well, by the AM-GM inequality. Then by (i) we know

$$I(a,b) = \frac{1}{a} K\left(\sqrt{1 - \frac{b^2}{a^2}}\right).$$

We similarly have

$$I\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{2}{a+b} K\left(\sqrt{1 - \frac{ab}{\left(\frac{a+b}{2}\right)^2}}\right) = \frac{2}{a+b} K\left(\frac{a-b}{a+b}\right).$$

So we require that

$$K\left(\sqrt{1 - \frac{b^2}{a^2}}\right) = \frac{2a}{a+b} K\left(\frac{a-b}{a+b}\right)$$

follows from the identity in #1550

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\,K(k)$$

when we set $k = (a-b)/(a+b)$.

(iv) Recall that given $a \geqslant b > 0$ we define sequences

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad a_1 = a,\ b_1 = b,$$

with the sequence a_n converging down to $\mathrm{agm}(a,b)$ and b_n converging up to $\mathrm{agm}(a,b)$. So by (iii) we have

$$I(a,b) = I(a_1, b_1) = I(a_2, b_2) = \cdots = I(a_n, b_n)$$

for each n. If we take the limit as $n \to \infty$ and use part (ii) we see

$$I(a,b) = I(\mathrm{agm}(a,b), \mathrm{agm}(a,b)) = \frac{\pi}{2\mathrm{agm}(a,b)}$$

and hence $\mathrm{agm}(a,b) I(a,b) = \pi/2$.