Solution (#1556) Let $a > 0$. From #1388 we know that

$$I(a) = \int_0^\infty \frac{\ln x}{x^2 + a^2}\,\mathrm{d}x = \frac{\pi \ln a}{2a}.$$

Differentiating under the integral sign, we find that

$$I'(a) = -2a \int_0^\infty \frac{\ln x}{(x^2 + a^2)^2}\,\mathrm{d}x = \frac{\pi}{2a^2}(1 - \ln a).$$

Hence

$$\int_0^\infty \frac{\ln x}{(x^2 + a^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3}(\ln a - 1).$$

Now if we set $x = u^2$ then we see that

$$\begin{aligned}
\int_0^\infty \frac{x^{1/2}\ln x}{(1+x)^2}\mathrm{d}x &= \int_0^\infty \frac{u \ln u^2}{(1+u^2)^2}\,2u\,\mathrm{d}u \\
&= 4\int_0^\infty \frac{u^2 \ln u}{(1+u^2)^2}\,\mathrm{d}u \\
&= 4\left\{\int_0^\infty \frac{\ln u}{1+u^2}\,\mathrm{d}u - \int_0^\infty \frac{\ln u}{(1+u^2)^2}\,\mathrm{d}u\right\} \\
&= 4\left\{I(1) - \int_0^\infty \frac{\ln u}{(1+u^2)^2}\,\mathrm{d}u\right\} \\
&= 4\left\{0 - \left(-\frac{\pi}{4}\right)\right\} = \pi.
\end{aligned}$$